Mathematical Reviews

Edited by

W. Feller **E. Hille** **H. Whitney**

J. V. Wehausen, ***Executive Editor***

Vol. 14, No. 10 **November, 1953** **pp. 935–1048**

TABLE OF CONTENTS

AUTHOR INDEX

(*Continued on cover 3*)

MATHEMATICAL REVIEWS

Published monthly, except August, by

The American Mathematical Society, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

The American Mathematical Society
The Mathematical Association of America
The Institute of Mathematical Statistics
The Edinburgh Mathematical Society
Société Mathématique de France
Matematisk Forening i København
Het Wiskundig Genootschap te Amsterdam
The London Mathematical Society
Polskie Towarzystwo Matematyczne
Unión Matemática Argentina
Indian Mathematical Society
Unione Matematica Italiana

Editorial Office

Mathematical Reviews, 80 Waterman St., Providence 6, R. I.

Subscriptions: Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to Mathematical Reviews. Subscriptions may be addressed to Mathematical Reviews, Lancaster, Pennsylvania, but should preferably be addressed to the American Mathematical Society, 80 Waterman St., Providence 6, R. I.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Scientific Research, Headquarters, Air Research and Development Command, U. S. Air Force. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in paragraph (d-2), section 34.40, P. L. and R. of 1948, authorized November 9, 1940.

Mathematical Reviews

Vol. 14, No. 10 NOVEMBER, 1953 Pages 935-1048

FOUNDATIONS

*Rosser, J. Barkley. **Logic for mathematicians.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. xiv+530 pp. $10.00.

This book falls into two parts of which the first, comprising chapters I–IX, contains a rather broad treatment of logic, adapted in an original way to the needs of mathematicians by the insertion of numerous applications. Almost every logical theorem of any importance is thus illustrated by an example, taken from some mathematical textbook. Of course the complexity of these applications increases with the construction of the logical building; they range from simple examples taken from Euclid to a treatment of open and closed sets in Hausdorff space as an application of the logic of classes. Ch. I is introductory. In Ch. II the statement calculus is treated from an intuitive point of view and the method of truth tables is introduced. Ch. III, "The use of names", stresses the difference between a thing and its name. In Ch. IV the statement calculus is treated by the axiomatic method, and the completeness theorem is proved. In Ch. V, "Clarification", it is stated as the aim of the book to set up a symbolic logic such that none of its principles is known to be invalid (no definition of this term is attempted) and from which the existing body of mathematics can be derived. Some remarks are made about the modes of reasoning which are allowed in metalogic, but no adequate description of the finitist point of view is given. Ch. VI contains the predicate calculus of first order. The axioms are given in closed form, so that the only derivation rule needed is modus ponens. Attention is given to restricted quantification: if α is a variable restricted to the objects satisfying $K(x)$, then $(\alpha)F(\alpha)$ is an abbreviation for $(x).K(x)\supset F(x)$. In Ch. VII equality is introduced, and in Ch. VII the notation $\iota xF(x)$ for (the x such that $F(x)$). Ch. IX contains the logic of classes. It begins with a discussion of the notion of class and of the Russell paradox. Various proposals for eliminating the paradox are examined (here the description of Brouwer's point of view on p. 203–204 is erroneous). The system adopted for the book is that of Quine's "New foundations" [Amer. Math. Monthly **44**, 70–80 (1937)]; consequently, many theorems are subject to conditions of stratification. The formal system is now more precisely described; here equality is defined in terms of class membership and a system of 12 axiom schemes is given (p. 212–213). Additional axioms are introduced in the following chapters; they will be mentioned below. A section is devoted to variables restricted to a fixed class Σ; here a division analogous to the hierarchy of types is obtained.—In the second part of the book the program of Frege, Russell and Whitehead, the building up of mathematics on the basis of logic, is taken up with all the simplifications which later research has made possible. Here also the author keeps in contact with mathematical literature, from which he borrows examples to the main theorems. Ch. X, "Relations and functions", begins with the theory of natural numbers Nn. In sharp contrast with the ample explanations of other fundamental concepts, Frege's definition is given without any comment. The principle of mathematical induction is proved for stratified conditions, and Peano's fourth axiom is introduced as Axiom scheme 13, equivalent to the axiom of infinity. The calculus of relations is developed. There is an interesting discussion of the ambiguity of the word "function" in mathematics. The author uses "function" in the sense of "many-one-relation". Order relations are treated: a section on equivalence relations leads over to Ch. XI on cardinal numbers NC. Of course it cannot be proved that any class is similar to the class of its unit subclasses; a class with this property is called cantorian. The arithmetic of cardinal numbers is treated and it is proved that $Nn\subset NC$; a class is called finite if its cardinal number is a Nn. Various forms of the induction principle are discussed. As to definition by induction it is argued that intuitively it defines the function value for any natural number, but not the function itself. A formal proof is given for the latter, under a stratification condition. No finite class can be similar to a proper subclass of itself, but the converse is proved only by making use of the axiom of choice (Ch. XIV). In the next section it is proved among other things that Nn is cantorian. Finally, the arithmetical properties of the cardinal number of the continuum are proved. Ch. XII is on ordinal numbers. In Ch. XIII, "Counting", the author observes that for any fixed natural number n the formula $A(n)$, $m(m\,\varepsilon\,Nn.0<m\leq n)\,\varepsilon\,n$, can be proved, but no proof of (n): $n\,\varepsilon\,Nn.\supset.A(n)$ is known. Thus he proposes the latter as Axiom scheme 14. It allows one to prove that every finite class is cantorian. Ch. XIV is on the axiom of choice. The equivalence of this axiom with Zermelo's well-ordering theorem and with various forms of Zorn's lemma is proved. The denumerable axiom of choice is proposed as Axiom scheme 15; it is used to prove that every infinite class is similar to a proper subclass of itself, and that the union of a denumerable class of denumerable classes is denumerable. In the concluding Ch. XV a reading program is given with the intention of showing that the notions introduced in the book suffice to develop the bulk of modern mathematics. As it was impossible to summarize the formal deductions which form the main content of the book, this review stresses disproportionately some controversial points. The author has succeeded in giving a modern treatment of the subject matter of "Principia mathematica" [2nd ed., 3 vols., Cambridge, 1925, 1927]; at the same time he has written a book of great didactic value. *A. Heyting* (Amsterdam).

Gericke, H. **Algebraische Betrachtungen zu den Aristotelischen Syllogismen.** Arch. Math. 3, 421–433 (1952).

Certain algebraic aspects of the syllogism are considered by regarding the A, E, I, and O propositions as two-termed Boolean relations. Valid syllogistic forms become elementary principles in the theory of relational sums and products, relative products, etc. *R. M. Martin.*

Izumi, Yosihisa. Remarques sur la notion de la perfection. Tôhoku Math. J. (2) **4**, 252–256 (1952).

The author gives, using notations of Tarski [Monatsh. Math. Phys. **37**, 361–404 (1930)], definitions of strong and weak completeness, decidability and contradictoriness. He then attempts to prove that if a set of formulae is contradictory it is strongly complete, that if it is strongly complete it is weakly complete and decidable and that if it is decidable it is weakly complete. *A. Rose* (Nottingham).

Myhill, John. On the interpretation of the sign '⊃'. J. Symbolic Logic **18**, 60–62 (1953).

Let S be a logical system in which the following axioms and rules are valid. $A, A\supset B \vdash B$; if $A \vdash B$, then $\vdash A\supset B$; $A\&B \vdash A$; $A\&B \vdash B$; $A, B \vdash A\&B$; $A\&B\supset C \vdash A\supset .B\supset C$. A strict implication $\prec$ is introduced, which needs only to satisfy the rules: If $A \vdash B$, then $\vdash A \prec B$; if $\vdash A \prec B$, then $\vdash A\supset B$. If we allow the use of an existential quantifier $(\exists r)$, where r is a propositional variable, and if we admit the rules: $\ldots A\text{---} \vdash (\exists r)(\ldots r\text{---})$; if $\ldots r\text{---} \vdash A$, then $(\exists r)(\ldots r\text{---}) \vdash A$, where A is any formula, submitted in the last rule to the restriction that it does not contain r, then $A\supset B$ is interdeducible with $(\exists r)(r\&((r\&A) \prec B))$. This suggests an interpretation of $\supset$ in, e.g., the systems of Heyting, Johansson, and Fitch. *A. Heyting.*

Wang, Hao. Certain predicates defined by induction schemata. J. Symbolic Logic **18**, 49–59 (1953).

A definition of a predicate, definable by induction (d.i.) in number theory Z is given, for which the definition of $M(n, k)$ [Hilbert and Bernays, Grundlagen der Mathematik, Bd. II (H-B II), Springer, Berlin, 1939, p. 337] is a characteristic example. It is shown that every predicate d.i. is explicitly expressible in an extension L_1 of Z. The extension L_1 contains, besides the signs of Z, variables $X, Y, \cdots$ for classes of natural numbers, with the corresponding quantifiers; its atomic sentences have the form $m \,\eta\, X$ (m belongs to X). L_1 has as additional axioms:

(Ax 1) $(X)((0 \,\eta\, X\&(m)(m \,\eta\, X\supset(m+1 \,\eta\, X))\supset(n)n \,\eta\, X)$;
(Ax 2) $m=n\supset(m \,\eta\, X\supset n \,\eta\, X)$;
(Ax 3) $(EY)(m)(m \,\eta\, Y\equiv p)$

for any sentence p of L_1 which does not contain Y. It follows that a consistency proof for Z can be given in L_1. As also the predicate $Q(c, a, n, k)$ of H.-B. II, p. 368 is d.i., Gentzen's consistency proof for Z is also formalizable in L_1. Analogous results are valid for a system given by Quine [J. Symbolic Logic 6, 139–149 (1941), pp. 140–145; these Rev. **3**, 289]. A predicate d.i. $P(n, k)$ is said to be definable* in a system L if for a certain expression P_1 in L the defining equation of P is provable for any given constant k. All predicates d.i. are definable* in a system L_2 which is weaker than L_1. The system L_2 contains the same signs and sentences as L_1, but as additional axioms only: (Ax′ 1) If φm is any sentence of L_2, then $m=n\supset(\varphi m\supset\varphi n)$; (Ax′ 2) If p is any sentence of Z, then $(EX)(m)(m \,\eta\, X\equiv p)$. The same holds for systems L_4 and L_5 which are parts of the Bernays system of set theory. In these systems an adequate truth definition for Z can be obtained, but no consistency proof. The author examines what additions to these systems are necessary for formalizing a consistency proof for Z. Extensions to other systems than Z, such as Zermelo set theory, are indicated. *A. Heyting* (Amsterdam).

Rose, Alan. Conditioned disjunction as a primitive connective for the erweiterter Aussagenkalkül. J. Symbolic Logic **18**, 63–65 (1953).

Church has conjectured [Portugaliae Math. **7**, 87–90 (1948); these Rev. **10**, 421] that conditioned disjunction, together with the universal and existential quantifiers, forms a complete self-dual set of independent connectives for the higher calculus of propositions (erweiterter Aussagenkalkül, not to be confused with the more familiar higher functional calculus). In the present paper the author proves this conjecture, as extended to an m-valued calculus. The place of the conditioned disjunction is taken by a connective with $m+1$ arguments which involves the J-functions of Rosser and Turquette [J. Symbolic Logic **10**, 61–82 (1945); these Rev. **7**, 185]. The same connective was used by the present author in an earlier paper [Math. Ann. **123**, 76–78 (1951); these Rev. **12**, 790] in order to prove a similar theorem for the ordinary m-valued calculus of propositions. *A. Robinson* (Toronto, Ont.).

Zykov, A. A. The spectrum problem in the extended predicate calculus. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 63–76 (1953). (Russian)

In the restricted predicate calculus with equality [called $P1$ in this review] the "spectrum" of a formula $\mathfrak{A}$ is the set $T(\mathfrak{A})$ of cardinal numbers such that $\mathfrak{A}$ holds in a field of objects M if and only if the cardinal of M is in $T(\mathfrak{A})$. It follows from Löwenheim's Theorem that only spectra whose members do not exceed $\aleph_0$ need be considered in $P1$. There is no analogue of Löwenheim's Theorem in the "extended calculus", $P2$, i.e., the predicate calculus with predicate- and object-quantifiers, and hence there is no universal upper bound for the sets $T(\mathfrak{A})$. The author considers that, on this account, the set-theoretic antimonies prevent the above definition of the spectrum from being taken over as it stands into $P2$; and that the way out by defining $T(\mathfrak{A}, \tau)$, for a given cardinal τ, as "the set of cardinals not exceeding τ such that . . . etc." is unsatisfactory. He is therefore preparing to give another definition. The present paper contains the proofs of three preparatory theorems (but not the definition). Theorem I. Every formula of $P2$ is "synonymous" with another in prenex form in which all predicate-quantifiers precede all object-quantifiers. Here "synonymous" means taking the same truth-value in all evaluations. The predicate-variable part of the prefix, briefly the p-prefix, consists of alternate blocks of consecutive $\exists$ and $\forall$ quantifiers. Theorem II says that if there are l of these blocks, there is a synonymous formula whose p-prefix contains l blocks, each containing only one quantifier. Theorem III. Let $\mathfrak{A}$ be a formula of $P2$ with no free variables, and l the maximum number of object-variable-places of any predicate-variable of $\mathfrak{A}$. Then there exists a prenex formula $\mathfrak{B}$ of $P2$, whose p-prefix is $\exists P.\forall Q$, where P and Q are respectively 1-place and 2-place predicate variables; and $\mathfrak{B}$ holds in fields of cardinal μ if and only if $\mu=\tau+\tau^l+\tau^{2^l}$, where $\mathfrak{A}$ holds in fields of cardinal τ. *M. H. A. Newman* (Manchester).

Curry, Haskell B. Mathematics, syntactics and logic. Mind **62**, 172–183 (1953).

This paper was presented to the International Congress for the Unity of Science [Chicago, September 1941] and is closely related to the author's earlier paper on "Some aspects of the problem of mathematical rigor" [Bull. Amer. Math. Soc. **47**, 221–241 (1941); these Rev. **2**, 340] and to his

"Outlines of a formalist philosophy of mathematics" [North-Holland Publishing Co., Amsterdam, 1951; these Rev. 13, 422]. The definition of mathematics is based on the notion of a formal system, and it is argued against Carnap that a formal system is not the same thing as a syntax. In a formal system we have only one language, while in a syntax we have two. This remark leads on to a discussion on pseudo-object sentences, with special reference to Carnap's "Logical syntax of language" [Harcourt, Brace, New York, 1937]. *E. W. Beth* (Amsterdam).

Kreisel, G. On the concepts of completeness and interpretation of formal systems. Fund. Math. 39 (1952), 103–127 (1953).

This paper contains comments on and additions to an earlier paper [J. Symbolic Logic 16, 241–267 (1951); 17, 43–58 (1952); these Rev. 14, 122, 440], to which reference shall be made by (I). The notion of interpretation is extended to interpretations of a formal system $\mathfrak{F}$ in another system F; in such an interpretation there is associated to every formula $\mathfrak{A}$ of $\mathfrak{F}$ a sequence of formulas A_n of F. Though the no-counter-example interpretation of Z_μ as given in (I) §39 can be replaced by an interpretation in which no function variables occur, function variables are indispensable in the following sense: There is no interpretation of Z_μ in a formal system F satisfying the following conditions: its variables are free individual variables, its predicates are decidable, its function symbols are computable, and from each A_n associated with $\mathfrak{A}$ the latter can be proved by the predicate calculus of first order. An interpretation is called complete if $\mathfrak{A}$ can be proved when some A_n is true. The interpretations of (I) §§21, 24, and 39 are proved to be complete. (The notion of truth must be suitably defined for F.) Finally, a complete, non-standard interpretation of a system equivalent to Z_1 of Hilbert and Bernays [Grundlagen der Mathematik, Bd. II, Springer, Berlin, 1939, p. 324] in full number theory is given; in this paragraph the reasoning is not finitist. *A. Heyting.*

Kreisel, Georg. Some concepts concerning formal systems of number theory. Math. Z. 57, 1–12 (1952).

The author discusses the equivalence of three consistency concepts for formal systems of arithmetic formed by the addition of verifiable free variable formulas as axioms to the system Z_μ of Hilbert-Bernays. The concepts considered are external consistency (with respect to the class of general recursive functions) [Hilbert and Bernays, Grundlagen der Mathematik, v. 2, Springer, Berlin, 1939, p. 282], the author's free variable interpretation [J. Symbolic Logic 16, 241–267 (1951); 17, 43–58 (1952); these Rev. 14, 122, 440], and omega consistency. The author gives an almost complete account of his no-counter-example interpretation for both the predicate calculus and the arithmetic systems. This treatment avoids some of the extremely complicated details of the earlier work (for a criticism of the earlier work see the remarks of Rosser [ibid. 18, 78–80 (1953)]). It is shown here that, for systems of arithmetic of the type described, external consistency with respect to the class of general recursive functions is equivalent to the existence of a no-counter-example interpretation. However, omega consistency is a more stringent requirement, for the author shows that there exists a formula of Z_μ of the form $(Ex)(y)(Ez)Q(x, y, z)$, with Q primitive recursive, such that the addition of this formula to Z_μ as an axiom results in a system externally consistent but not omega consistent. *D. Nelson* (Washington, D. C.).

Yablonskiĭ, S. V. On superpositions of functions of the algebra of logic. Mat. Sbornik N.S. 30(72), 329–348 (1952). (Russian)

The problem is to find necessary and sufficient conditions that a given finite set of expressions of the classical 2-valued propositional calculus form a basis for all truth-functions; more precisely, conditions that from m expressions $\Phi_r(A_1, A_2, \cdots, A_n)$, each in n variables, all truth-functions are obtainable by combinations of the operations (1) "superposition", i.e., replacement of any occurrence of a variable by an expression already obtained, and (2) replacement of $A_1, A_2, \cdots, A_n$ by any other set of variables $B_1, B_2, \cdots, B_n$, not necessarily all distinct. Call an expression, Φ, "of type α" if $\Phi(A_1, A_2, \cdots, A_n) \neq \neg\Phi(\neg A_1, \neg A_2, \cdots, \neg A_n)$ (the inequality meaning that they have not the same truth-table); of type β if it is not expressible (in the same sense) with the help of $\equiv$ alone; of type γ if $\Phi(T, T, \cdots, T) = F$; of type δ if $\Phi(F, F, \cdots, F) = T$; of type ϵ if it is not expressible by means of & and $\vee$ alone. The main theorem states that the expressions $\Phi_1, \Phi_2, \cdots, \Phi_m$ form a basis if and only if they include at least one of each of the types $\alpha, \beta, \gamma, \delta, \epsilon$; and if the base is minimal, $m \leqq 4$. The proof is long and complicated, and cannot be summarised here. *M. H. A. Newman* (Manchester).

Bereczki, Ilona. Lösung eines Markovschen Problems betreffs einer Ausdehnung des Begriffes der elementaren Funktion. Acta Math. Acad. Sci. Hungar. 3, 197–218 (1952). (Russian summary)

The author has shewn [C. R. Premier Congrès Math. Hongrois, 1950, Akadémiai Kiadó, Budapest, 1952, pp. 409–417] that not all primitive-recursive functions are "elementary", i.e., obtainable from constants and variables by the "four rules": addition, subtraction ($|x-y|$), multiplication, division ($[x/y]$); together with the summation- and product-operations ($\sum^n$ and $\prod^n$). This is now generalised by considering the class of functions formed analogously to the elementary functions, but using an arbitrary finite class of primitive-recursive functions instead of the four rules, and a finite set of primitive-recursive functional operations for $\sum^n$ and $\prod^n$. The result still holds. *M. H. A. Newman* (Manchester).

Martin, R. M. On truth and multiple denotation. J. Symbolic Logic 18, 11–18 (1953).

Combining the method of multiple denotation [R. M. Martin and J. H. Woodger, same J. 16, 191–203 (1951); these Rev. 13, 310] with Quine's method of framed ingredients [Mathematical logic, Harvard Univ. Press, 1951, p. 291–305; these Rev. 13, 613] the author obtains a simple definition of truth for a formalization of Zermelo set theory without using recursive definitions. In its simplest form it comes to the following (for the exact definition of the notions which are here intuitively used the paper must be consulted). A sentence c is true, if some prenex normal form a for c, beginning with a universal quantifier, is true. If $a = (x)(\text{---}x\text{---})$, then a is true if there is an expression b denoting just those objects x such that ---x---, and at the same time denoting everything. It is indicated how the same method can be applied to the simplified theory of types. *A. Heyting* (Amsterdam).

Skolem, Th. A remark on a set theory based on positive logic. Norske Vid. Selsk. Forh., Trondheim 25 (1952), 112–116 (1953).

Russell's paradox shows that 2-valued logic is not a suitable framework for an intuitive set theory. But, as shown

here, it is not possible to build a satisfactory intuitive set theory even in a system without any negation at all. We take as axioms of intuitive set theory: 1) If $\phi(x)$ is a propositional function, then $\hat{x}\phi(x)$ is a set and $(y \varepsilon \hat{x}\phi(x)) \equiv \phi(y)$; 2) $(x=y) \equiv (z)((x \varepsilon z) \rightarrow (y \varepsilon z))$. Identity of sets is defined by extensionality: $(x=y) \equiv (z)((z \varepsilon x) \equiv (z \varepsilon y))$. Using only the positive predicate calculus [Hilbert and Bernays; Grundlagen der Mathematik, vol. 1, Springer, Berlin, 1934, p. 66], a first order predicate calculus based on conjunction, disjunction, and implication, it is shown that in any such set theory all sets must be identical, and the theory therefore trivial. To do this, write 'Λ' for '$\hat{x}(y)(x \varepsilon y)$', '$V$' for '$\hat{x}((x \varepsilon x) \supset (x \varepsilon x))$', '$\dot{V}$' for '$\hat{x}((x \varepsilon x) \supset (x \varepsilon \Lambda))$'. Hence $(\dot{V} \varepsilon \dot{V}) \equiv ((\dot{V} \varepsilon \dot{V}) \rightarrow (\dot{V} \varepsilon \Lambda))$, which on simplification shows that $\dot{V} \varepsilon \Lambda$. So by definition of Λ, $(y)(\dot{V} \varepsilon y)$. If we now write '$\{m\}$' for '$\hat{x}(x=m)$', then $\dot{V} \varepsilon \{m\}$, so $V=m$; hence $m=n$, for any m, n. It is noted that the logic used here is weaker than intuitionistic logic. It should be pointed out, however, that in such a system of set theory every proposition P expressible in the system is also provable in it. This is done using Curry's paradox: Write 'S' for '$\hat{x}((x \varepsilon x) \supset P)$'. Then $(S \varepsilon S) \supset ((S \varepsilon S) \supset P)$, hence $(S \varepsilon S) \supset P$. Thus $S \varepsilon S$, and so P.

I. L. Novak Gál (Ithaca, N. Y.).

Löb, M. H. Concatenation as basis for a complete system of arithmetic. J. Symbolic Logic 18, 1–6 (1953).

Using a limited kind of universal quantification in place of the proper ancestral and concatenation in place of the ordered-pair function as primitive operators, complete systems for arithmetic are constructed equivalent with Myhill's K [same J. 15, 185–196 (1950); these Rev. 12, 579].

R. M. Martin (Philadelphia, Pa.).

Ryll-Nardzewski, C. The role of the axiom of induction in elementary arithmetic. Fund. Math. 39 (1952), 239–263 (1953).

Following Skolem, we may interpret the notion of "a property" which occurs in the principle of mathematical induction for the positive integers as a predicate of one variable which can be expressed in the lower functional calculus in terms of the primitive relations and functors (some or all of the notions of equality, order, sum, product, etc.). Writing down the principle of induction successively for all these predicates, we obtain a sequence of axioms. If we add the usual rules of arithmetic and order, there results an infinite set of axioms for the system of positive integers. The present author establishes the important result that an equivalent finite set of axioms for the positive integers does not exist in the lower functional calculus, even if new functors are added as primitive notions. An important part in the proof is played by the non-normal models of axiomatic systems for positive integers whose existence was first established by Skolem. Given any finite system of axioms in the lower functional calculus, the author constructs a predicate of one variable which is inductive although it is not satisfied by all the elements of a certain non-normal model for the positive integers, as mentioned. The existence of such a predicate proves the main result.

A. Robinson (Toronto, Ont.).

Mostowski, A. On models of axiomatic systems. Fund. Math. 39 (1952), 133–158 (1953).

This paper applies a discussion of various notions of models and their interrelationships to a study of the following question: Given a formal system S based on an infinite number of axioms $A_1, A_2, \cdots$, is it possible to prove in S the consistency of the system s based on a finite number $A_1, \cdots, A_n$ of these axioms? There is a real model of the first kind of s in S if a translation of the conjunction of the axioms of s is provable in S. If S is a system of set theory, we have a real model of the second kind of s in S if the translations of all the axioms of s are satisfied in S. If S contains a system of arithmetic of the positive integers, we have a real model of the third kind of s in S if we can prove in S that the translations of all axioms of s are valid. The different kinds of models are closely connected with various theorems on finite axiomatizability, ω-consistency, and mutual interpretability. Among the theorems proved are the following. I. If the system S of the arithmetic of real numbers is consistent and s is a finitely axiomatizable subsystem of S, then the sentence $N(S)$ expressing in S the consistency of s is provable in S. II. The arithmetic of real numbers is not finitely axiomatizable. III. If S is the system of arithmetic of positive integers based on Peano's axioms, then S is not finitely axiomatizable. [This was first proved by a different method by C. Ryll-Nardzewski in the paper reviewed above.]

I. Novak Gál (Ithaca, N. Y.).

Rasiowa, H. A proof of the compactness theorem for arithmetical classes. Fund. Math. 39 (1952), 8–14 (1953).

The methods of proof used in a paper by H. Rasiowa and R. Sikorski [Fund. Math. 37, 193–200 (1950); these Rev. 12, 661] are here adapted to give a mathematical proof of the compactness theorem for arithmetical classes: If K is a set of arithmetical classes and $\bigcap_{X \varepsilon K} X = 0$, then there is a finite set $L \subseteq K$ such that $\bigcap_{X \varepsilon L} L = 0$. [See A. Tarski, Proc. Internat. Congress Math. Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R. I., 1952, pp. 705–720; these Rev. 13, 521.]

I. Novak Gál (Ithaca, N. Y.).

Myhill, John. Criteria of constructibility for real numbers. J. Symbolic Logic 18, 7–10 (1953).

In this paper, two theorems and a conjecture announced previously [same J. 15, 185–196 (1950); these Rev. 12, 579] are proved and shown equivalent with results of Specker [ibid. 14, 145–158 (1949); these Rev. 11, 151]. These are concerned with the definability of expressions for real numbers within certain kinds of constructivistic systems.

R. M. Martin (Philadelphia, Pa.).

Kurepa, Đuro. Proof of the principle of total induction. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke 277, 238–248 (1950). (Serbo-Croatian)

This is a partly expository article on the inductive property of the positive integers. The argument given is ingenious and reasonably brief, but it is not convincing to the reviewer on account of the extreme vagueness of the axiom system for the positive integers which is employed. The author states, however, that his argument is valid for either of two standard axiom systems for the positive integers.

E. Hewitt (Seattle, Wash.).

ALGEBRA

★**Birkhoff, Garrett, and Mac Lane, Saunders. A survey of modern algebra.** Rev. ed. Macmillan Co., New York, N. Y., 1953. xi+472 pp. $6.50.

For a review of the first edition [1941] see these Rev. 3, 99. Although some additions and rearrangements have been made for this edition, the content remains essentially the same.

Greenwood, R. E. The number of cycles associated with the elements of a permutation group. Amer. Math. Monthly 60, 407–409 (1953).

A derivation is given for the generating function enumerating permutations of n by number of cycles; namely

$$\sum_1^n c(n,k)x^k = x(x+1)\cdots(x+n-1).$$

This is a special case of results given by Touchard [Acta Math. 70, 243–297 (1939)]. From this the mean and variance of the corresponding probability variable $c(n,k)/n!$ are obtained with results in agreement with those obtained more simply by Feller [Bull. Amer. Math. Soc. 51, 800–832 (1945); these Rev. 7, 128]. *J. Riordan.*

Parker, W. V. The matrices AB and BA. Amer. Math. Monthly 60, 316 (1953).

If the field has sufficiently many elements, the conclusion $\det(\lambda I - AB) = \det(\lambda I - BA)$, valid for square matrices A, B, can be proved for the case when A, B are both singular by a continuity argument. The starting point is the lemma that if λ, y are indeterminates, the identity

$$\det(\lambda I - (A - Iy)B) = \det(\lambda I - B(A - Iy))$$

holds, since equality is seen to hold for sufficiently many y. Then replace y by 0. *J. L. Brenner* (Pullman, Wash.).

Katz, Leo, and Olkin, Ingram. Properties and factorizations of matrices defined by the operation of pseudo-transposition. Duke Math. J. 20, 331–337 (1953).

The authors define the pseudo-transpose (briefly p-transpose) C^0 of an $n\times n$ matrix C by the relation $C^0 = JC'J$, where C' is the ordinary transpose of C and $J = I_p \dotplus (-I_q)$, $p+q=n$. Using this generalised definition of transposition, many well-known properties of orthogonal, symmetric and skew matrices are established for p-orthogonal ($C^0 = C^{-1}$), p-symmetric ($C^0 = C$) and p-skew ($C^0 = -C$) matrices. In particular, analogues are obtained of the Cayley factorisation of a real orthogonal matrix and of the Toeplitz factorisation of a positive definite symmetric matrix. *D. E. Rutherford* (St. Andrews).

Tornheim, Leonard. The Sylvester-Franke theorem. Amer. Math. Monthly 59, 389–391 (1952).

The author gives a simple, direct, and elegant proof of the following Sylvester-Franke Theorem: If $A^{(s)}$ is an sth compound of the n by n matrix A, then $|A^{(s)}| = |A|^e$, where $|A|$ is the determinant of A and $e = {}_{n-1}C_{s-1}$. The proof is as follows. (a) If B is any matrix such that $|B^{(s)}| = |B|^e$, and if B is transformed into C by elementary transformations, then $|C^{(s)}| = |C|^e$. (b) There exists a matrix D such that $|D^{(s)}| = |D|^e$, and such that D can be transformed into A by elementary transformations. Similar methods can be used to establish three corollaries which give properties of compound matrices. [For other proofs of the Sylvester-Franke Theorem, see G. B. Price, same Monthly 54, 75–90 (1947); these Rev. 8, 366.] *G. B. Price.*

Littlewood, D. E. On unitary equivalence. J. London Math. Soc. 28, 314–322 (1953).

Given two $n\times n$ matrices A, B, the author considers the problem of finding an $n\times n$ matrix X such that the relations $XAX^* = B$, $XX^* = I$ hold, where I is the identity matrix. The method used is analogous to that of the reviewer [Acta Math. 86, 297–308 (1952); these Rev. 13, 717], who had solved the more general problem $XA_iX^* = B_i$, $XX^* = I$, $i = 1, 2, \cdots$. Some of the complications which can occur in the solution are explained by means of an example. *J. L. Brenner* (Pullman, Wash.).

Schwerdtfeger, H. Problems in the theory of matrices and its applications. Australian J. Sci. 15, 112–115 (1953).

Charles, Bernard. Sur la permutabilité des opérateurs linéaires. C. R. Acad. Sci. Paris 236, 1722–1723 (1953).

The author studies the problem of determining conditions under which two commuting matrices A and B are expressible as polynomials in a third matrix. Over an infinite field the problem can be reduced to the case where A and B are jointly indecomposable. Then, over a field of characteristic 0, a sufficient condition is given in terms of inclusion relations between certain subspaces attached to A and B. *I. Kaplansky* (Chicago, Ill.).

Charles, Bernard. Un critère de maximalité pour les anneaux commutatifs d'opérateurs linéaires. C. R. Acad. Sci. Paris 236, 1835–1837 (1953).

The author exhibits certain conditions which suffice to ensure the maximality of a commutative algebra of matrices. It is asserted that these types account for all examples up to dimension 6. *I. Kaplansky.*

Charles, Bernard. Un exemple général d'anneau commutatif d'opérateurs linéaires tel que $R''\neq r(R, I)$. C. R. Acad. Sci. Paris 236, 2027–2029 (1953).

R denotes a commutative algebra of linear transformations, R'' its double commutator, and $r(R, I)$ the algebra generated by R and the identity matrix. The author gives a class of algebras for which the indicated inequality holds; R is assumed to be a direct sum, with certain properties when the summands act on the vector space. An explicit example is given with R a 3-dimensional trivial algebra and the summands 1- and 2-dimensional. *I. Kaplansky.*

Newton, R. H. C. On quasi-commutable infinite matrices. Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. 15, 120–130 (1953).

Let $A, B, \cdots$ be infinite matrices. The author calls the matrix $C^{(r)}$ defined by $C^{(r)} = AC^{(r-1)} - C^{(r-1)}A$, $r = 2, 3, \cdots$, $C^{(1)} = AB - BA$, the r-commutator of A and B. The matrices A and B are called r-commutable if $C^{(r)} = 0$ (this notion is asymmetric in A and B for $r>1$). The possibility of generalization of some properties of commutable matrices to r-commutable matrices is discussed. As examples we quote: (1) A well-known lemma about matrices which commute with $A^{-1}DA$, where D is a diagonal matrix with distinct elements, cannot be generalized for $r>1$; (2) if the matrices A, B, r-commute, the corresponding methods of summation (provided they are regular) are consistent for all bounded sequences. *G. G. Lorentz* (Detroit, Mich.).

Gyires, B. Verallgemeinerung eines Determinantensatzes von J. Hunyady. Publ. Math. Debrecen 2, 290–291 (1952).

For $1\leqq k\leqq n-1$, let $C_k(A)$ be the kth derived matrix of the $n\times n$ matrix A; let A^* be the transpose of A. Then

$$\det C_k(A)\cdot\det (C_{n-k}(A)\pm C_{n-k}(A^*))$$
$$=\det (C_{n-k}(A))\cdot\det (C_k(A)\pm C_k(A^*)).$$

If k is 1 and $\det A\neq 0$, this reduces to

$$\det \operatorname{adj} (A\pm A^*)=(\det A)^{n-2}\det (A\pm A^*),$$

as announced by Hunyady [Nouvelles Ann. Math. (3) 1, 384 (1882)]. *J. L. Brenner* (Pullman, Wash.).

Clark, F. E. A sufficient condition for positivity of polynomial forms. Proc. Amer. Math. Soc. 3, 988–992 (1952).

In this paper, a distribution over a finite, partially ordered set $S=\{F_i\}$ is the assignment of a real number f_i, called the supply, to each $F_i\in S$. If $F_j\leqq F_i$ and $f_j<0<f_i$, then changes in the supplies to f_i-g_{ij} at F_i, f_j+g_{ij} at F_j are considered, where $0\leqq g_{ij}\leqq \min (f_i, -f_j)$. If all the supplies can be made non-negative by such changes, the distribution is called adequate. A reserve of a distribution is the total supply of a set which contains with F_j all F_i such that $F_j\leqq F_i$. Elementary methods show that a distribution is adequate if and only if all of its reserves are non-negative. This theorem is applied in combination with Muirhead's Theorem [Hardy, Littlewood, and Pólya, Inequalities, Cambridge, 1934, Theorem 45] to give a sufficient condition that a symmetric form be non-negative or positive. *H. W. Kuhn.*

Gonçalves, J. Vicente. Remarque sur le calcul abrégé d'un résultant. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) 1, 403–404 (1951).

Abstract Algebra

Büchi, J. Richard. Representation of complete lattices by sets. Portugaliae Math. 11, 151–167 (1952).

Let L be a complete lattice and $\cup$, $\cap$, $\subset$ and (a_ν) the symbols for the sup, inf, the partial ordering and the subset of L consisting of the elements $a_\nu\in L$. Let $\mathfrak{N}$ denote a class of subsets of L including all one-element-subsets. The lattice L is called $\mathfrak{N}$-representable if there exists an one-to-one mapping of L on a system of sets S which transforms the inf in L into set-intersections and the sup of subsets which are elements of $\mathfrak{N}$ into set-unions. An element $u\in L$ is called $\mathfrak{N}$-subirreducible if for any element $(a_\nu)\in\mathfrak{N}$ the relation $u\subset\bigcup_\nu a_\nu$ implies the existence of an element $a_\nu\in(a_\nu)$ including u. A subset B of L is called a U-basis of L if every element $a\in L$ is the sup of a subset of B. L satisfies the $\mathfrak{N}$-subdecomposition property if the $\mathfrak{N}$-subirreducible elements of L form a U-basis of L.

Some results: L is $\mathfrak{N}$-representable if and only if it satisfies the $\mathfrak{N}$-subdecomposition property. (In the formulation of the Theorem 15 read "$\mathfrak{N}$-subdecomposition" and "$\mathfrak{N}$-subirreducible" instead of "$\mathfrak{N}$-decomposition" and "$\mathfrak{N}$-irreducible".) Every complete chain satisfies the strongest distributive laws [G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 146, formula (22′); these Rev. 10, 673]. The strongest distributive laws in L don't imply the isomorphism of L with the system of all subsets of a set [see G. N. Raney, Proc. Amer. Math. Soc. 3, 677–680 (1952); these Rev. 14, 612]. The economy of representations and the system of all U-bases of L are studied. Theorem 19 is false. (Counter-exemple: Let L be the class of all subsets of the set containing exactly three points 1, 2, 3 and let sup and inf be set-union and set-intersection. Further let $\mathfrak{N}$ denote the class consisting of all one-point-subsets of L and of the set (a_ν), $\nu=1, 2$, where $a_1=(1, 3)$, $a_2=(2, 3)$. Then L is $\mathfrak{N}$-representable and for $b=(1, 2)$ the set $(b\cap a_\nu)$, $\nu=1, 2$, fails to belong to $\mathfrak{N}$. Therefor the distributive law $d_\mathfrak{N}$ does not hold.) *M. Novotný.*

Petresco, Julian. Théorie relative des chaînes. IV. Normalités de Schreier et de Zassenhaus. C. R. Acad. Sci. Paris 236, 2029–2031 (1953).

[For previous papers of this series, see same C. R. 235, 226–228, 1087–1089 (1952); 236, 651–653 (1953); these Rev. 13, 901; 14, 346, 612.] The author considers a relation N between pairs of elements of a lattice, especially elements of two given chains or of chains derived from them as in the first paper cited. He announces various necessary and sufficient conditions on N for conditions of the Schreier-Zassenhaus type (without consideration of isomorphism) to hold. [There appear to be misprints, whose rectification is difficult in the absence of proofs.] *P. M. Whitman.*

Lyapin, E. Systems with an infinite operation. Doklady Akad. Nauk SSSR (N.S.) 50, 49–51 (1945). (Russian)

Let φ be a mapping of an ordered set $\mathfrak{A}$ of indices into a given set $\mathfrak{M}$; the set $\mathfrak{A}$ considered together with the map φ is called a word of the set $\mathfrak{M}$; the length of the word is the cardinal of $\mathfrak{A}$. Given a transfinite number α, the collection of all words of $\mathfrak{M}$ with lengths less than $\aleph_\alpha$ is denoted by $\mathfrak{W}^\alpha(\mathfrak{M})$. An operation in $\mathfrak{M}$ is a law determining a collection of so-called admissible words of $\mathfrak{M}$ such that to each admissible word W is associated a non-empty subset of $\mathfrak{M}$ called the composite of W and denoted by $[W]$. A set together with an operation defined in it is called a system. If $\mathfrak{H}$ is a subset of a system $\mathfrak{S}$, the properties of the subsystem $\{\mathfrak{H}\}$ generated by $\mathfrak{H}$ and a transfinite construction for $\{\mathfrak{H}\}$ are studied. In a system $\mathfrak{S}$, let $\mathfrak{S}^*$ be the collection of all elements not belonging to any composite of a word of $\mathfrak{S}$. A system $\mathfrak{S}$ belongs to the class Ω^α provided $\mathfrak{W}^\alpha(\mathfrak{S})$ is the set of all admissible words of $\mathfrak{S}$; a system $\mathfrak{S}$ of class Ω^α belongs to the class Γ^α provided $\{\mathfrak{S}^*\}=\mathfrak{S}$ and every element of $\mathfrak{S}$ belongs to the composite of no more than one word. A transfinite construction is given, yielding all systems of class Γ^α; furthermore, if α and the cardinals $\mathfrak{m}$ and $\mathfrak{n}$ are prescribed, a system $\mathfrak{S}$ in Γ^α exists for which $\mathfrak{S}^*$ has cardinal $\mathfrak{n}$ and the cardinal of the composite of every admissible word is $\mathfrak{m}$. Every subsystem of a system of class Γ^α belongs to Γ^α. A mapping φ of a system $\mathfrak{S}$ onto a system $\mathfrak{H}$ is called a perfect homomorphism provided, for every word W of the system $\mathfrak{S}$, $[\varphi W]=\varphi[W]$ (where a natural extension of notation interprets the symbol φW). To each system of class Ω^α there is a system of class Γ^α which has a perfect homomorphism onto the given system. *R. A. Good.*

Rédei, L. Über die Determinantenteiler. Acta Math. Acad. Sci. Hungar. 3, 143–150 (1952). (Russian summary)

Let R be a commutative ring with unit element and let M be a matrix with coefficients in R. Let $\mathfrak{D}_k$ be the ideal generated in R by the determinants of order k extracted from M. Then it follows from the theory of elementary divisors that, when R is a Dedekind ring, we have $\mathfrak{D}_k^2\supset\mathfrak{D}_{k-1}\mathfrak{D}_{k+1}$ (where $\mathfrak{D}_0=(1)$). The author proves that, for any ring R,

$m(k)\mathfrak{D}_{k-1}\mathfrak{D}_{k+1}\subset\mathfrak{D}_k^2$, where $m(k)$ is the L.C.M. of the integers $1, \cdots, k$. The proof depends on a new identity on determinants, which is of the following form. Let D be a determinant of order $2k$, d a minor of order $k+1$ of D and Δ its complementary minor (taken with its sign); then $d\Delta$ is expressible as a linear combination of products of minors of order k by their complementary minors (with rational but not integral coefficients). A counterexample shows that it is not always true that $\mathfrak{D}_{k-1}\mathfrak{D}_{k+1}\subset\mathfrak{D}_k^2$. In the statement of theorem 2, line 7 of the statement, $k+1$ should be replaced by $k-1$. *C. Chevalley* (Nagoya).

Rédei, L. Vollidealringe im weiteren Sinn. I. Acta Math. Acad. Sci. Hungar. 3 (1952), 243–268 (1953). (Russian summary)

A ring is a vollidealring (v-ring) if every submodule of the ring is an ideal, while a ring is a v-ring in the extended sense (e-v-ring) if each of its subrings is an ideal. The author characterized all v-rings in a previous paper [Monatsh. Math. 56, 89–95 (1952); these Rev. 14, 127]. This paper characterizes all e-v-rings that are generated by a single element. If R is such an infinite e-v-ring, then necessarily R is of the form $xI[x]/x(x-a)\mathfrak{a}$, where I is the ring of integers and $\mathfrak{a}$ is a product of ideals of the form $(x(x-b), p)$, $p\nmid b$, or of the form $(x-b, p)$ taken over distinct prime factors p of a. Similar but more complicated formulas give all the finite e-v-rings generated by a single element.

R. E. Johnson (Northampton, Mass.).

Sato, Hazimu. Zum Teilerkettensatz in kommutativen Ringen. Proc. Japan Acad. 29, 10–12 (1953).

If $\mathfrak{R}$ is an arbitrary commutative ring, an ideal $\mathfrak{H}$ of $\mathfrak{R}$ is said to be "half-prime" if the ring $\mathfrak{R}/\mathfrak{H}$ has no non-zero nilpotent elements. The main theorem of this paper then runs as follows. Let $\mathfrak{R}$ be a commutative ring. The following two conditions are then equivalent to the ascending chain condition for ideals of $\mathfrak{R}$ (i.e., that $\mathfrak{R}$ be Noetherian): (1) if $\mathfrak{H}_1\subseteq\mathfrak{H}_2\subseteq\cdots$ is an ascending chain of "half-prime" ideals, then all of them must be the same from some point on; (2) if an ideal $\mathfrak{A}$ possesses one and only one prime ideal divisor $\mathfrak{P}$ with the property that between $\mathfrak{A}$ and $\mathfrak{P}$ there is no prime ideal of $\mathfrak{R}$, then the ascending chain condition prevails for ideals between $\mathfrak{A}$ and $\mathfrak{P}$. Given a chain of ideals $\mathfrak{A}_1\subseteq\mathfrak{A}_2\subseteq\cdots$ the author makes several uses of the associated ascending chain of "half-prime" ideals $\mathfrak{H}_1\subseteq\mathfrak{H}_2\subseteq\cdots$ where $\mathfrak{H}_i=\{x\in\mathfrak{R}\,|\,x^m\in\mathfrak{A}_i$ for some integer $m\}$.

I. N. Herstein (Chicago, Ill.).

Seidenberg, A. A note on the dimension theory of rings. Pacific J. Math. 3, 505–512 (1953).

An integral domain O is said to be n-dimensional if there exists in O a chain of n prime ideals, all different from the zero and the unit ideal, but no such chain with more than n members. The author is concerned with the truth or falsity of the statement: (*) If O is n-dimensional, then $O[x]$ is $(n+1)$-dimensional (x being an indeterminate). It is known that (*) is true for Noetherian rings, and the author gives a new proof of this fact. He shows that it need not be true in general, but that the dimension of $O[x]$ will in all cases be between $n+1$ and $2n+1$, endpoints allowed. If (*) is false for an integral domain O, then there exists a minimal prime ideal p in O such that (*) is false either for O_p or for O/p. Thus if (*) is false, there must exist a 1-dimensional ring for which it is false, and the question is to some extent reduced to the 1-dimensional case. If O is 1-dimensional, then (*) is true for O if and only if it is true for the integral closure of O (which is also 1-dimensional). If O is 1-dimensional and integrally closed, then (*) is true if and only if O is a multiplication ring, i.e., every quotient ring of O with respect to a prime ideal is a valuation ring. This result is a consequence of the following theorem, known in the case of a finite discrete principal order, but new in the generality given here: If O is integrally closed, p prime in O, a an element of the quotient field of O, but a non-$\varepsilon\, O_p$, a^{-1} non-$\varepsilon\, O_p$, then $O[a]p$ is prime in $O[a]$, contracts to p in O, and has a residue class ring isomorphic to a polynomial ring over O/p. *I. S. Cohen* (Cambridge, Mass.).

Kasch, Friedrich. Ein Satz über den Endomorphismenring eines Vektorraums. Arch. Math. 3, 434–435 (1952).

A theorem of Jacobson [Trans. Amer. Math. Soc. 57, 228–245 (1945); these Rev. 6, 200] asserts that a subring of the full ring of endomorphisms Σ of a vector space Ω over a division ring H is a dense ring of transformations over Ω if and only if it is two-fold transitive. In the present paper the author proves the following generalization: If A is a simply transitive ring of linear transformations of Ω and D a two-fold transitive ring of linear transformations of Ω, then every non-zero A-D submodule of Σ (i.e., a left A module and right D module) as a set of transformations on Ω is dense. The author then exhibits two corollaries to this result.

I. N. Herstein (Chicago, Ill.).

Osima, Masaru. Notes on basic rings. Math. J. Okayama Univ. 2, 103–110 (1953).

The author generalizes the theory of basic rings, due to R. Brauer. This theory runs as follows. Let A be a ring with unit element and minimum condition on left ideals, R its radical. An idempotent e is chosen such that eAe/eRe is a direct sum of division rings. The ring eAe is called a basic ring for A; and A is determined up to isomorphism by eAe and the integers giving the size of the matrix rings in A/R. The author extends this theory to rings satisfying the following assumption, weaker than the minimum condition: A is a direct sum of a finite number of indecomposable left ideals, and this decomposition is unique up to A-isomorphism.

I. Kaplansky (Chicago, Ill.).

van der Kulk, W. On polynomial rings in two variables. Nieuw Arch. Wiskunde (3) 1, 33–41 (1953).

Etude des automorphismes de l'anneau de polynomes $A=k[x, y]$ à deux variables sur un corps: ceux ci sont produits de transformations linéaires et de transformations de la forme $(x, y)\rightarrow(x, y+f(x))$ $(d^0(f)\geqq 2)$. Cette étude équivaut à celle des transformations birationnelles et partout birégulières du plan affine P. Par une telle transformation, et si celle-ci n'est pas linéaire, les droites de P sont transformées en courbes unicursales ayant un seul point à l'infini, et ce point leur est commun; cette remarque est à la base d'un procédé de réduction des degrés qui fournit le résultat annoncé. La décomposition des automorphismes de A est essentiellement unique modulo les transformations linéaires. Il est aussi montré que (lorsque k n'est pas le corps à 2 éléments) la quotient du groupe des automorphismes de A par son groupe des commutateurs est isomorphe au groupe multiplicatif k^* au moyen de l'application $s\rightarrow D(s(x), s(y))/D(x, y)$. *P. Samuel.*

Northcott, D. G. Hilbert's function in a local ring. Quart. J. Math., Oxford Ser. (2) 4, 67–80 (1953).

L'auteur généralise la théorie des polynômes caractéristiques de Hilbert au cas d'un idéal I d'un anneau local R:

étant donné un idéal Q primaire pour l'idéal maximal de R il pose:

$$\chi_Q(r, I) = L(Q^r/Q^{r+1}) - L((I \cap Q^r)/(I \cap Q^{r+1}))$$

($L(M)$ désignant la longueur du module M). Par passage à R/I on voit que $\chi_Q(r, I) = P_{\bar{Q}}(r+1) - P_{\bar{Q}}(r)$, où $\bar{Q} = (Q+I)/I$, et où $P_{\bar{Q}}(n)$ est la fonction $L((R/I)/\bar{Q}^n)$ introduite par le rapporteur [J. Math. Pures Appl. (9) **30**, 159–205, 207–274 (1951); ces Rev. **13**, 980]. L'auteur retrouve et précise certains résultats du rapporteur, et démontre les principales propriétés de $\chi_Q(r, I)$ par passage à l'anneau gradué associé à R filtré par les (Q^n) (ou plutôt un anneau de polynômes $\bar{R}$ sur R/Q dont celui ci est un quotient), et par transposition à $\bar{R}$ de la méthode de van der Waerden [Math. Ann. **99**, 497–541 (1928)] pour l'étude des polynômes de Hilbert: pour r assez grand $\chi_Q(r, I)$ est un polynôme en r dont le degré est la dimension d de l'idéal I; notons $\mathrm{ord}_Q(I)n^d/d!$ son terme de plus haut degré. Lorsque R est un anneau local régulier, M son idéal maximal, et $(a_1, \cdots, a_s)$ des éléments de R dont les formes initiales dans $\bar{R}$ engendrent un idéal de rang s, le polynôme $\chi_M(r, (a_1, \cdots, a_s))$ est calculé: son degré est $d-s$ et l'on a $\mathrm{ord}_M((a_1, \cdots, a_s)) = \prod_{i=1}^{s} n_i$ où n_i est l'entier défini par $a_i \in M^{n_i}$, a_i non-$\in M^{n_i+1}$.

P. Samuel (Clermont-Ferrand).

Behrens, Ernst-August. Assoziativ auflösbare Ringe. Math. Z. **58**, 25–40 (1953).

Soit G le groupe des cycles (non nécessairement homogènes) de l'espace projectif P_n. Grâce à la définition des multiplicités d'intersection pour toutes composantes, propres et excédentaires [cf. Samuel, J. Math. Pures Appl. (9) **30**, 159–205, 207–274 (1951); ces Rev. **13**, 980; et Behrens, Math. Z. **55**, 199–215 (1952); ces Rev. **13**, 981], un "produit d'intersection" est partout défini dans G; mais, à cause des composantes excédentaires, celui-ci n'est pas associatif. Cependant l'ensemble G_q des cycles de dimension $<q$ est un idéal de G, et G_{q+1}/G_q est un anneau associatif. Ceci introduit la notion générale d'anneau associativement résoluble (ass. rés.): A est ass. rés. s'il existe une chaîne finie de sous anneaux A_q de A telle que A_q soit idéal de A_{q+1} et que A_{q+1}/A_q soit associatif. Tout quotient et tout sous anneau d'un anneau ass. rés. est ass. rés. Un sous anneau d'une somme directe d'anneaux simples est associatif dès qu'il est ass. rés. Le radical d'un anneau ass. rés. est étudié, ainsi que son groupe d'automorphismes. Des résultats sont donnés sur la construction d'un anneau ass. rés. à partir des quotients A_{q+1}/A_q. Enfin l'auteur caractérise les sous anneaux associatifs de l'anneau G des cycles de P_n.

P. Samuel (Clermont-Ferrand).

Almeida Costa, A. Über die unterdirekten Modulnsummen. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **2**, 115–160 (1952).

This is largely an expository account of topics connected with subdirect sums. The basic definitions and results are given for modules, whereas the literature largely concerned itself with the more special case of rings. The section headings are as follows: subdirect sums, special subdirect sums, discrete direct sums, subdirectly irreducible modules, the Jacobson radical, semi-simple modules, modules over a division ring, the G-submodule of Brown and McCoy.

I. Kaplansky (Chicago, Ill.).

Lamprecht, Erich. Allgemeine Theorie der Gaussschen Summen in endlichen kommutativen Ringen. Math. Nachr. **9**, 149–196 (1953).

Let R be a finite commutative ring with unit element. A Gaussian sum attached to R is defined to be an expression of the form $\tau = \sum_{x \in M} \chi(x) e(x)$, where M is the multiplicative group of invertible elements of M, χ a character of M, and e a character of the additive group of R. The paper is divided in two parts: a) computation of all Gaussian sums attached to R; b) reduction of various types of Gaussian sums to Gaussian sums attached to finite commutative rings.

A finite commutative ring R with unity may be represented as a direct composite of rings R_i which are primary (R is called primary when its zero divisors are all nilpotent, i.e., when there is only one prime ideal in R; this prime ideal will be denoted by $\mathfrak{p}$). Correspondingly, any Gaussian sum attached to R decomposes into a product of Gaussian sums attached to the rings R_i. Assume now that R is primary, and let $\mathfrak{q}$ be the ideal of those $x \in R$ such that $e(xy) = 1$ for all $y \in R$. If the condition $x \equiv 1 \pmod{\mathfrak{q}}$ does not imply $\chi(x) = 1$, then it is shown that $\tau = 0$. If, however, $\chi(x)$ depends only on the residue class of $x \bmod \mathfrak{q}$, then τ may be considered as a Gaussian sum attached to $R/\mathfrak{q}$, multiplied by a scalar. This reduces the problem to the case where $\mathfrak{q} = \{0\}$. In that case, R cannot be an arbitrary primary ring; it satisfies the following two conditions, equivalent to each other: 1) $\{0\}$ is not representable as an intersection of ideals all $\neq \{0\}$; 2) there exists a character e_0 of the additive group of R such that every other character is of the form $x \to e_0(ax)$ for some fixed a. Such rings are called irreducible, and an e_0 which satisfies 2) is called "echt".

Assume now that R is irreducible and that e is "echt". Then χ is called "eigentlich" if there is no ideal $\mathfrak{q}' \neq \{0\}$ such that $\chi(x)$ depends only on the residue class of x modulo $\mathfrak{q}'$. Then it is proved that there always exist "eigentliche" characters and that $\tau = 0$ when χ is not "eigentlich", while, if χ is "eigentlich", then $|\tau|^2$ is the number of elements of R. The determination of the actual value of τ is reduced to the computation of Gaussian sums attached to the field $R/\mathfrak{p}$, a problem previously solved by Stickelberger. In order to accomplish this reduction, the structure of R has to be analyzed very precisely. It had already been proved by Krull that, under the assumptions made, the operation of passing from an ideal $\mathfrak{q}$ to its annihilator $\{0\} : \mathfrak{q}$ gives a one-to-one mapping of order 2 of the set of ideals of R onto itself. A Jordan-Hölder sequence $(\mathfrak{q}_0 = R, \mathfrak{q}_1, \cdots, \mathfrak{q}_m = \{0\})$ of ideals of R is called "invariant by inversion" if, for each i, $\{0\} : \mathfrak{q}_i$ belongs to the sequence. Two cases have to be distinguished according as there exists such a sequence or not, and the first case has to be subdivided further in two, according as m is even (in which case we set $\mathfrak{q} = \mathfrak{r} = \mathfrak{q}_{m/2} = \{0\} : \mathfrak{q}$) or odd (in which case we set $\mathfrak{q} = \mathfrak{q}_{(m-1)/2}$, $\mathfrak{r} = \mathfrak{q}_{(m+1)/2} = \{0\} : \mathfrak{q}$). If there is no Jordan-Hölder sequence invariant by inversion, then the author proves that m is even, that $R/\mathfrak{p}$ is not of characteristic 2, and that there exists a Jordan-Hölder sequence for which $\mathfrak{q}_i = \{0\} : \mathfrak{q}_{m-i}$ for $i \leq (m-2)/2$; we then set $\mathfrak{q} = \mathfrak{q}_{(m-2)/2}$, $\mathfrak{r} = \mathfrak{q}_{(m+2)/2} = \{0\} : \mathfrak{q}$. In any one of these cases, the sum τ is computed by grouping together those terms corresponding to those $x \in M$ which belong to one and the same residue class mod $\mathfrak{r}$ and observing that the mapping $x \to \chi(1+x)$ $(x \in \mathfrak{r})$ gives an additive character of $\mathfrak{r}$, which may be written in the form $e(ax)$ for some suitably selected a in R. The final results of the computation are too complicated to be reported here; they are in the form of explicit expressions of τ in terms of Gaussian sums for $R/\mathfrak{p}$

and of various parameters depending on χ and on the structure of R. In the case where $R/\mathfrak{p}$ is of characteristic 2, the results also involve an undefined "formal trace expression" in $K=R/\mathfrak{p}$ with values integers mod 4. This apparently amounts to the following: there is a function Q on K with values integers mod 4 which satisfies the functional equation $Q(x+x')=Q(x)+Q(x')+2\,\mathrm{Tr}\,xx'$, where Tr is the absolute trace in K and the elements of the prime subfield of K are considered as integers mod 2. The existence of such a function may be established by making use of a base of K over its prime subfield.

Generalized Gaussian sums occur both in the theory of algebraic number fields and in that of fields of algebraic functions of one variable over finite fields. The author reduces these two kinds of Gaussian sums to Gaussian sums attached to finite rings, which allows one to compute them completely. In both cases, the main step of the reduction is to express the given sums as products of sums each attached to some place of the field under consideration. The author also indicates that it is possible to attach Gaussian sums to any 0-dimensional ideal in a polynomial ring over a finite field and to rings of Witt vectors over a finite field; the latter are a special case of the local sums attached to Gaussian sums in algebraic number fields. *C. Chevalley.*

Kuroš, A. G. On the theory of locally simple and locally central algebras. Ukrain. Mat. Žurnal 3, 205–210 (1951). (Russian)

An algebra is said to be locally finite if every finitely generated subalgebra is finite-dimensional; it is locally simple if every finite subset can be embedded in a finite-dimensional simple subalgebra; the terms "locally central simple" and "locally matricial" have analogous definitions. Theorem 1 asserts that over an arbitrary field there exists a locally finite central simple algebra which is not locally central simple. This answers in the negative a problem which had been open for some time. The construction is worth recording. The desired algebra V is the union $\bigcup V_n$ of an ascending sequence of subalgebras, where V_n is a direct sum of two full matrix algebras of degrees s_n and t_n, and we have $(s_n, t_n)=1$, $s_{n+1}=s_n+t_n$, $t_{n+1}=s_n+2t_n$. (For instance, one may take $s_1=t_1=1$ and define s_n, t_n inductively.) The embedding of V_n in V_{n+1} is the (more or less) obvious one, subject to the requirement that they have a common unit element.

A locally central simple algebra is said to be primary (for the prime p) if every finite-dimensional simple subalgebra has order a power of p. It is known that a locally central simple algebra of countable order is a direct product of primary algebras. Answering a question raised by Kuročkin [Mat. Sbornik N.S. 22(64), 443–454 (1948); these Rev. 10, 8], the author shows in Theorem 2 that this is false without countability. The construction begins with an algebra A given as the direct product of full matrix algebras of degree p, one for each prime. It is known that A admits an isomorphism θ onto a proper subalgebra with the same unit element. We now form $\bigcup A_n$, where each A_n is isomorphic to A, and the embedding of A_n in A_{n+1} is accomplished via the isomorphism θ. This is shown to furnish the desired counter-example.

In the final section, the author raises the following question: can every locally finite algebra be embedded in a locally matricial one? A footnote added in proof reports that Kuročkin has found a counter-example. But in Theorem 3 it is shown that the answer is affirmative for locally simple algebras of countable order. *I. Kaplansky.*

Koszul, Jean-Louis. Sur les représentations linéaires des algèbres de Lie résolubles. C. R. Acad. Sci. Paris 236, 2371–2372 (1953).

Let $\mathfrak{g}$ be a Lie algebra over a field k, let M be a finite-dimensional representation module of $\mathfrak{g}$ over k and let $H(\mathfrak{g}, M)$ be the cohomology group of $\mathfrak{g}$ operating on M. A cohomology class c of $H(\mathfrak{g}, M)$ is called "effaçable" if there exists a finite-dimensional representation module N of $\mathfrak{g}$ over k such that N contains M and c becomes trivial in $H(\mathfrak{g}, N)$. The author gives a brief sketch of the proof of the following result: A Lie algebra over a field of characteristic 0 is solvable if and only if every cohomology class of degree >0 in $H(\mathfrak{g}, M)$ is effaçable for any representation module M of $\mathfrak{g}$. *K. Iwasawa* (Cambridge, Mass.).

Hochschild, G., and Serre, J-P. Cohomology of Lie algebras. Ann. of Math. (2) 57, 591–603 (1953).

In what follows, G denotes a Lie algebra over a field F, M a G-module, and K a subalgebra of G. The set of elements of M annihilated by G is denoted by M^G. The cohomology groups of G in M are denoted by $H^n(G, M)$; the relative cohomology groups of G modulo K in M are denoted by $H^n(G, K, M)$. The group of n-cochains of G in M is denoted by A^n, and we set $A=\sum_n A^n$ (direct). The authors introduce a filtration (A_j) in A as follows: A_j is the sum of its intersections with the A^n, and $A_j\cap A^n$ is $\{0\}$ if $j>n$, while, if $j\leqq n$, it consists of those cochains which, as n-linear mappings of G^n into M, take the value 0 when $n-j+1$ of their arguments are in K. This filtration defines a spectral sequence whose groups will be denoted by $E_r^{j,i}$. The E_1-term of this sequence may be computed in the most general case. But, as usual, the interesting results come from the determination of the E_2-term, and this determination is possible only under certain supplementary assumptions, which are of two types: either one supposes that K is an ideal, or one makes assumptions of reductivity on K.

Assume first that K is an ideal. Then $H^n(K, M)$ has a natural structure of G/K-module, and $E_2^{j,i}$ is then isomorphic to $H^j(G/K, H^i(K, M))$. Once this is established, consequences may be derived by using the same methods as in an earlier paper [Trans. Amer. Math. Soc. 74, 110–134 (1953); these Rev. 14, 619] by the same authors; actually, the reader is referred to this paper for the proofs which are not carried out in the present one. These consequences consist in the setting up of two exact sequences relative to the case where some cohomology groups of K in M vanish (it would be too long to give here the detailed statements of these exact sequences). Moreover, it is proved that, if G/K is of dimension p and K of dimension q, then $H^{p+q}(G, M)$ is isomorphic to $H^p(G/K, H^q(K, M))$.

The other type of results (when K is not an ideal) is an extension of the results of Koszul [Bull. Soc. Math. France 78, 65–127 (1950); these Rev. 12, 120]. Assume that F is of characteristic 0, G finite-dimensional, K reductive in G, and M semi-simple for K. Then $E_2^{j,i}$ is isomorphic to $H^i(K, F)\otimes H^j(G, K, M)$ (where F is considered as a trivial K-module). Assume furthermore that the restriction homomorphism maps $H^n(G, F)$ onto $H^n(K, F)$ for every $n\geqq 0$; then $H(G, M)$ $(=\sum_n H^n(G, M))$ is isomorphic as a graded module to $H(K, F)\otimes H(G, K, M)$ (with its total degree). On the other hand, if L is an ideal of G such that G/L is semi-simple, then $H(G, M)$ is isomorphic to $H(G/L, F)\otimes H(L, M)^G$ ($H(L, M)$ having a natural structure of G-module). *C. Chevalley* (Nagoya).

Aškinuze, V. G. A theorem on the splittability of J-algebras. Ukrain. Mat. Žurnal 3, 381–398 (1951). (Russian)

The author proves the Wedderburn principal theorem for Jordan algebras of characteristic 0 (if R is the radical of A, there exists a subalgebra S such that A is the vector space direct sum of R and S). The method is to prove that all possible simple summands A/R can be lifted to subalgebras of A, and heavy use is made of Albert's techniques as well as of his classification of the simple algebras. Independently, the same theorem was proved by Penico [Trans. Amer. Math. Soc. 70, 404–420 (1951); these Rev. 12, 798]. Though the two proofs are similar, there are certain technical differences. *I. Kaplansky* (Chicago, Ill.).

Wright, Fred B. Absolute valued algebras. Proc. Nat. Acad. Sci. U. S. A. 39, 330–332 (1953).

Soit A une algèbre à division non associative sur les réels, munie d'une valeur absolue avec les axiomes habituels, et ayant un élément unité. L'auteur montre d'abord que, muni de la norme $|x|$, A est isomorphe à un espace préhilbertien, en prouvant que pour $|x|=|y|=1$, on a $|x-y|^2+|x+y|^2\geqq 4$. Munissant A du produit scalaire (x, y) tel que $(x, x)=|x|^2$, il montre ensuite que tout élément x de A peut s'écrire $x=a+by$, où a et b sont réels, y orthogonal à 1 et tel que $y^2=-1$, d'où l'existence d'un conjugué $x^*=a-by$, et le fait que tout élément de A est algébrique de degré $\leqq 2$ sur R. Les résultats d'Albert sur les algèbres à division non associatives montrent alors que A est isomorphe au corps des réels, ou au corps des complexes, ou au corps des quaternions, ou à l'algèbre de Cayley-Dickson. *J. Dieudonné* (Evanston, Ill.).

Dürbaum, Hansjürgen. Zur Theorie der nichtkommutativen Bewertungen. Proc. Amer. Math. Soc. 4, 418–422 (1953).

Schilling [Bull. Amer. Math. Soc. 51, 297–304 (1945); these Rev. 6, 201] has proved the following: if F is a field relatively complete in a valuation V, and A is a division algebra over F such that any two elements generate a finite-dimensional subalgebra, then in an extension of V to A the value group must be commutative. The author observes that relative completeness of F is unnecessary, and indeed the theorem admits an extension to the case of any pair of division rings A, F such that A is algebraic over F in a weak sense. Two further theorems of this kind are given. The problem is reduced to one on ordered groups, and use is made of a theorem of B. H. Neumann [Amer. J. Math. 71, 1–18 (1949); these Rev. 10, 428]: if in an ordered group some power of x commutes with y, then x commutes with y. A similar method was used by Zelinsky [Bull. Amer. Math. Soc. 54, 175–183 (1948); these Rev. 9, 408] in his investigation of the non-associative case. *I. Kaplansky.*

Berezanskiĭ, Yu. M., and Kreĭn, S. G. Hypercomplex systems with a compact basis. Ukrain. Mat. Žurnal 3, 184–204 (1951). (Russian)

Detailed exposition of results already announced [Doklady Akad. Nauk SSSR (N.S.) 72, 5–8, 237–240 (1950); these Rev. 12, 188, 189]. The authors have changed the term "continuous algebra" to the one given in the title. *I. Kaplansky* (Chicago, Ill.).

Kustaanheimo, Paul, and Qvist, Bertil. On differentiation in Galois fields. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 137, 12 pp. (1952).

Järnefelt, G. On finite approximation of solutions of two ordinary differential equations belonging to the classical quantum mechanics. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 138, 23 pp. (1952).

The formal properties of differentiation in finite fields are discussed in the first of these two papers. In the second, application is made to consider polynomial approximations to differential equations over a finite field. No particular difficulties are encountered in these investigations. It would seem to the reviewer more in the spirit of this sequence of papers to consider difference equations rather than differential equations. *Marshall Hall* (Columbus, Ohio).

Theory of Groups

Thierrin, Gabriel. Quelques propriétés des sous-groupoïdes consistants d'un demi-groupe abélien D. C. R. Acad. Sci. Paris 236, 1837–1839 (1953).

A complex H of a demi-group D is called consistent [P. Dubreil, Mém. Acad. Sci. Inst. France (2) 63, no. 3 (1941); these Rev. 8, 15] if $ab \in H$ implies $a \in H$ and $b \in H$. The author studies the consistent subgroupoids of an abelian demi-group D. They constitute a complete semi-lattice T under inclusion. T is a lattice if D contains a neutral (=identity) element. T is a Boolean algebra if D is gaussian (i.e., every element of D has a unique factorization into primes). *A. H. Clifford* (Baltimore, Md.).

Thierrin, Gabriel. Sur une équivalence en relation avec l'équivalence réversible généralisée. C. R. Acad. Sci. Paris 236, 1723–1725 (1953).

If H is a complex of a demi-group D, and $a \in D$, the right compound ("composé à droite") F_a of H by a is defined to be the set of all elements x of D such that $a \in Hx$. Define $a\omega_H b$ $(a, b \in D)$ to mean that either $F_a \cap F_b \neq 0$ (0=empty set) or else $F_a=F_b=0$, and let Ω_H be the transitive closure of ω_H. Then Ω_H is an equivalence relation in D which is regular on the right ($a\Omega_H b$ implies $ac\Omega_H bc$) within HD. If $H=S$ is a subgroupoid of D such that $SD=D$, then Ω_S coincides with the generalized right reversible equivalence Σ_S defined by P. Dubreil [Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 10, 183–200 (1951); these Rev. 14, 12]. A complex H of D is called astricted on the right ("astreint à droite") if $Hx \cap Hy \neq 0$ implies $Hx=Hy$. Every coset Ha is then an equivalence class mod Ω_H, and the converse holds except for the class $V_H=D-HD$. Each class X is also astricted on the right, and Ω_X coincides with Ω_H within XD. A number of other results are stated. No proofs are given. *A. H. Clifford* (Baltimore, Md.).

Chehata, C. G. On an ordered semigroup. J. London Math. Soc. 28, 353–356 (1953).

The author shows that not every fully (i.e., linearly) ordered semigroup can be embedded in a group. He does this by introducing an order relation into Malcev's example [Math. Ann. 113, 686–691 (1937)] of a semigroup which cannot be so embedded. *A. H. Clifford.*

Nakada, Osamu. Partially ordered abelian semigroups. II. On the strongness of the linear order defined on abelian semigroups. J. Fac. Sci. Hokkaido Univ. Ser. I. 12, 73–86 (1952).

Let S be a partially ordered abelian semigroup (p.o. semigroup) as defined in I of the series [same J. 11, 181–189 (1951); these Rev. 13, 817]. For a, $b \in S$, define aTb to mean $a^m = b^n$ for some positive integers m, n. T is an equivalence relation in S, and the equivalence class $T(a)$ containing a is called the complete sector defined by a. The author discusses the logical interrelations among nine properties applicable to S, of which we state seven (a, b, c are elements of S). (α) $a^n = b^n$ for some n implies $a = b$. (γ) If $ab = b$ for some a, $b \in S$, then a is the unit element of S. (δ) $ac = bc$ implies aTb. Cancellation: $ac = bc$ implies $a = b$. (A) $a^2 \geqq a$ implies $ab \geqq b$ for all $b \in S$, and dually. Strength: $ac \geqq bc$ implies $a \geqq b$. Normality: $a^n \geqq b^n$ for some n implies $a \geqq b$. Strength implies cancellation implies (γ) and (δ); strength also implies (A); normality implies (α). If S is an (unordered) abelian semigroup, then (α), (γ), and (δ) imply cancellation and are necessary and sufficient that a strong linear order be definable in S. If S is linearly ordered, cancellation implies all the other eight properties. Other implications and 14 examples are given.

If a is a positive element (i.e., $a^2 \geqq a$) of a normal p.o. semigroup S, then the complete sector $T(a)$ is order-isomorphic with a subsemigroup of the additive semigroup of all non-negative rational numbers. A p.o. semigroup S satisfying (A) can contain at most one idempotent e, and, if so, e is the unit element of S; a is positive if and only if $a \geqq e$, and dually; if S does not contain an idempotent, we can adjoin one without losing property (A).

A. H. Clifford (Baltimore, Md.).

Ellis, David. Remarks on isotopies. Publ. Math. Debrecen 2, 175–177 (1952).

A groupoid $G(^*)$ is generalized Abelian if there exist four permutations P, Q, R, S of $G(^*)$ such that $xP^*yQ = yR^*xS$ for all x, y in $G(^*)$. The generalized commutative law, like the generalized associative law of T. Evans [J. London Math. Soc. 25, 196–201 (1950); these Rev. 12, 75] is preserved under isotopy. A quasigroup is isotopic to an Abelian group if and only if it is generalized Abelian and generalized associative in the sense of Evans. Two semilattices are isotopic if and only if they are isomorphic. *D. C. Murdoch.*

***Ledermann, Walter. Introduction to the theory of finite groups.** 2d ed. Oliver and Boyd, Edinburgh and London; Interscience Publishers, Inc., New York, 1953. viii+160 pp. \$1.55.

For the contents of the first edition of this text see these Rev. 10, 427. The changes in the second edition are not extensive. They consist mainly in the revision of the section on the isomorphism theorems and the rewriting of the chapter on Abelian groups. The latter is now written in additive notation and includes the basis theorem for infinite Abelian groups with a finite number of generators.

D. C. Murdoch (Vancouver, B. C.).

Richert, Hans-Egon. Über die Anzahl Abelscher Gruppen gegebener Ordnung. II. Math. Z. 58, 71–84 (1953).

Let $a_0(n)$ denote the number of essentially different abelian groups of order n. In a previous paper [Math. Z. 56, 21–32 (1952); these Rev. 14, 349] the author used the van der Corput-Phillips theory of exponent pairs to improve an estimate of $A_0(x) = \sum_{n \leq x} a_0(n)$ for large x, obtained by D. G. Kendall and the reviewer [Quart. J. Math., Oxford Ser. 18, 197–208 (1947); these Rev. 9, 226]. He now applies the same method to a related problem of Šapiro-Pyateckiĭ [Mat. Sbornik N.S. 26(68), 479–486 (1950); these Rev. 12, 316] who showed that, for large x and $(q, l) = 1$,

$$A_0(x; q, l) = \sum_{\substack{n \leq x \\ n \equiv l \pmod q}} a_0(n) = c_1(q)x + O(x^{\frac{1}{2}+\epsilon} q^{\frac{1}{2}}).$$

This the author replaces by the sharper result

$$A_0(x; q, l) = c_1(q)x + c_2(q)x^{1/2} + c_3(q)x^{1/3} + O(x^{3/10} q^{3/5} \log^{9/10} x).$$

The method of proof is similar but necessarily more complicated. *R. A. Rankin* (Birmingham).

Kertész, A. On fully decomposable abelian torsion groups. Acta Math. Acad. Sci. Hungar. 3, 225–232 (1952). (Russian summary)

The author has recently [same Acta 3, 121–126 (1952); these Rev. 14, 617] given a criterion for an Abelian p-group to be the direct sum of cyclic groups. In the present paper he extends his method to cover also the case of quasi-cyclic groups (of type (p^∞)). It is well known that among the Abelian p-groups only the cyclic and the quasi-cyclic groups are directly indecomposable so that the direct sums of groups of these two types exhaust the fully decomposable Abelian p-groups. Two definitions are required: A principal system of an Abelian p-group G is a maximal independent system P in which no element can be replaced by one of greater height in G without destroying the independence. The elements of infinite height in G are divided into two classes: a is of inner infinite height if the equation $p^t x = a$ has for each natural number t a solution $x \in G$ which is itself an element of infinite height; otherwise a is of outer infinite height. Then the following criterion holds: An Abelian p-group G is fully decomposable if and only if 1) G possesses a principal system and 2) G contains no elements of outer infinite height. As corollaries the author obtains a number of known results or their generalisations. (i) If in an Abelian p-group G the heights of the elements of finite height are bounded, then G is fully decomposable. (ii) If every non-zero element of an Abelian torsion-group is of infinite height, then the group is the direct sum of quasi-cyclic groups. (iii) An Abelian p-group G is fully decomposable if and only if 1) it contains no elements of outer infinite height, and 2) it is the union of a countable ascending chain $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ where for each n the heights of the elements of finite height in A_n are bounded. This generalises Kulikov's theorem [Mat. Sbornik 16(58), 129–162 (1945); these Rev. 8, 252]. (iv) A countable Abelian p-group G is fully decomposable if and only if each element of infinite height in G is contained in a subgroup of type (p^∞). This is Prüfer's classical result [Math. Z. 17, 35–61 (1923)].

K. A. Hirsch (London).

Fuchs, L. The direct sum of cyclic groups. Acta Math. Acad. Sci. Hungar. 3, 177–195 (1952). (Russian summary)

The author develops new criteria for an Abelian group to possess a basis, i.e., to be the (restricted) direct sum of cyclic groups. These are based on the new concept of the "relative" order of two elements of a group. Let G be an additively written Abelian group and a, b two of its elements of infinite order. Let S be a linearly independent set of elements of G which includes b. Then the phrase "a is of greater order than b, relative to S" shall mean that there exists a relation

$ra = sb + \sum_i s_i b_i$ ($b_i \in S$, $b \neq b_i$) in which $|r| > |s|$. (If b has finite order, the meaning shall be the usual.) Note that this order relation need not be transitive, and that there may be incomparable elements. The two criteria are now: 1. A subset B of the Abelian group G is a basis for G if and only if (i) it is a maximal independent system for G; (ii) the set is no longer independent if any element of B is replaced by one of greater order, relative to B. 2. A subset B of the Abelian group G is a basis for G if and only if (i) it is a (minimal) set of generators for G, not including 0; (ii) the set no longer generates G if any element of B is replaced by one of smaller relative order. (In the case of primary Abelian groups the first criterion has recently been established by A. Kertész [same Acta 3, 121–126 (1952); these Rev. 14, 617].)

As an application the author gives a new proof of Kulikov's result [Mat. Sbornik N.S. 16(58), 129–162 (1945); these Rev. 8, 252] that the existence of a basis is hereditary in subgroups. Finally, he proves the following new theorem: If the Abelian group G is an extension of H by F, and if both F and H possess bases, then G itself possesses a basis if and only if the following condition is satisfied: (i) if H is mixed or torsion-free, then the elements of the periodic part of F are bounded; (ii) if H is a torsion-group, then for all primes p which occur as orders in H the p-primary components of F have bounded orders. *K. A. Hirsch* (London).

Szele, T. On non-countable abelian p-groups. Publ. Math. Debrecen 2, 300–301 (1952).

A well-known counter-example in the theory of non-countable primary Abelian groups is the unrestricted direct sum of cyclic groups of order p^n, $n = 1, 2, \cdots$. The periodic part G of this group is p-primary and has the power of the continuum. It has no elements of infinite height and yet is not the direct sum (restricted or unrestricted) of cyclic groups. The author gives a new proof. Let in an assumed direct decomposition of G the sum of the summands of order $\leqq p^n$ be A_n. Then A_n turns out to be finite. But G is the union of the ascending chain $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ and would therefore be at most countable. *K. A. Hirsch* (London).

Szele, T. On direct sums of cyclic groups with one amalgamated subgroup. Publ. Math. Debrecen 2, 302–307 (1952).

The author considers the Abelian group G generated by elements $a_1, a_2, \cdots$, subject to the defining relations $m_1a_1 = m_2a_2 = \cdots = c$, where the m_i are integers greater than 1. He shows that G is the direct sum $G = A + B$ of a group A which is countable (or 0) and of the same type as G, but with distinct m_i, while B is the direct sum of finite cyclic groups. Hence he restricts himself to the study of groups G of type A. Such a group G is a direct sum of cyclic groups if and only if the set of integers m_i is bounded. The torsion-group T of G is a direct sum of cyclic groups and its torsion-free factor-group G/T is of rank 1 and isomorphic to the additive group of rational numbers generated by the reciprocals of the m_i. The group G is torsion-free if and only if all the m_i are co-prime in pairs. T is a direct summand of G if and only if there is no prime number of which arbitrarily high powers divide some m_i. If the additional relation $mc = 0$ is introduced, with $m > 0$, then the group is a direct sum of cyclic groups if and only if for no prime factor of m arbitrarily high powers divide some m_i.
K. A. Hirsch (London).

Mills, W. H. On the non-isomorphism of certain holomorphs. Trans. Amer. Math. Soc. 74, 428–443 (1953).

The author has shown recently [same Trans. 71, 379–392 (1951); these Rev. 13, 530] that two finite abelian groups with isomorphic holomorphs are isomorphic, while non-isomorphic non-abelian groups may possess isomorphic holomorphs, e.g., the dihedral and dicyclic groups of order $4n$ for $n \geqq 3$. The main result of the present paper is that if one of two groups with isomorphic holomorphs is finite and abelian, the other is abelian as well and hence, if finite, isomorphic to the first. This result disproves an assertion of G. A. Miller [Amer. J. Math. 21, 287–338 (1899)] that the cyclic group of order 8 and the octic group have isomorphic holomorphs. Actually the holomorph of the former is of order 32, that of the latter of order 64. Some of the author's additional results are: a new proof of the fact that the holomorph of an abelian group of odd order has only inner automorphisms, and a proof that the holomorph of an abelian p-group is indecomposable. *K. A. Hirsch.*

Barskaya, S. On the construction of primitive solvable groups. Ukrain. Mat. Žurnal 3, 61–84 (1951). (Russian)

This paper continues the work of D. Suprunenko (Souprounenko) on primitive soluble substitution groups [Mat. Sbornik N.S. 20(62), 331–350 (1947); these Rev. 8, 562]. It uses the same notations, definitions, and methods [polynomials and matrices over Galois fields]; in particular, a knowledge of the concepts of a primary and of a general substitution group and of the meaning of the integers m, r, s will be assumed in this review.

Suprunenko [loc. cit.] had given a complete classification of all primary soluble groups of degree p^q, where q is a prime number. The present author considers the case of a primitive soluble substitution group of degree p^{qt}, where q and t are distinct prime numbers. Since a maximal normal Abelian subgroup H of a general primary soluble group leads again to a decomposition of the form $qt = mrs$, H must belong to one of the following six possible types: 1) $m = qt$, $s = r = 1$; 2) $m = q$, $s = t$, $r = 1$; 3) $m = 1$, $s = qt$, $r = 1$; 4) $m = 1$, $s = q$, $r = t$; 5) $m = q$, $s = 1$, $r = t$; 6) $m = s = 1$, $r = qt$. In the first four cases the construction can be reduced to that of primary soluble groups of degree p^q or p^t, i.e., the case dealt with by Suprunenko. The present paper gives a complete solution for the case 6, and a further paper (which has not yet appeared in print) will deal with case 5, thus completing the classification. *K. A. Hirsch* (London).

Mařík, Jan. The Verlagerung of a group into its subgroups. Časopis Pěst. Mat. 76, 23–34 (1951). (Czech)

This paper contains a detailed explanation of the first, and partly the second, paragraph of the last chapter, entitled "Verlagerung in eine Untergruppe", of H. Zassenhaus's "Lehrbuch der Gruppentheorie" [Teubner, Leipzig-Berlin, 1937]. The original text of the above paragraphs is rather difficult. The author adds some simple applications of the theory, e.g., the proof that there is no simple group of order 144. *O. Borůvka* (Brno).

Itô, Noboru. On Π-structures of finite groups. Tôhoku Math. J. (2) 4, 172–177 (1952).

Let N be a normal subgroup of a finite group G with index prime to its order; Theorem S (for Schur) states that there exists in G at least one complemented subgroup of N; Hypothesis Z (for Zassenhaus) conjectures that all complemented subgroups of N are conjugate to one another. The

author maintains that various results by Čunihin and Hall and Grün may, on the basis of these two propositions, be improved, extended, or more elegantly presented. The theory by Čunihin, who studied various Π-properties of groups for a set Π of primes [cf. Doklady Akad. Nauk SSSR (N.S.) **77**, 973–975 (1951); these Rev. **12**, 800; and the references there cited], is revised in a study of Π-soluble groups and Π-faithful groups. The theory by Hall [cf. Proc. London Math. Soc. (2) **43**, 507–528 (1937); and the references there cited] is generalized to Π-soluble groups of various types. Finally, the theory by Grün [cf. Math. Nachr. **1**, 1–24 (1948); these Rev. **10**, 504; and the references there cited] is elaborated to the corresponding Π-properties.
R. A. Good (College Park, Md.).

Smirnov, D. M. On groups of automorphisms of soluble groups. Mat. Sbornik N.S. **32**(74), 365–384 (1953). (Russian)

The paper contains the detailed proofs of the results which have been announced in two preceding notes [Doklady Akad. Nauk SSSR (N.S.) **76**, 643–646 (1951); **84**, 891–894 (1952); these Rev. **12**, 587; **14**, 14]. *K. A. Hirsch.*

Higman, Graham. On a problem of Takahasi. J. London Math. Soc. **28**, 250–252 (1953).

Let the group G be the inverse limit of a set of groups G_α. It is proved in this note that if each G_α is (i) locally free, and (ii) contains no infinite properly ascending sequence of n-generator subgroups for any fixed positive n, then G has the same properties. Any free group satisfies (i) and (ii) [Takahasi, J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **1**, 65–70 (1950); these Rev. **13**, 9]. Consequently, any inverse limit of free groups, as well as any subgroup of such a limit, has these properties. But the unrestricted free product of a countable set of free cyclic groups is an inverse limit of free groups and has countable subgroups which are not free [G. Higman, J. London Math. Soc. **27**, 73–81 (1952); these Rev. **13**, 623] demonstrating the existence of groups satisfying (i) and (ii) which are not free. *D. G. Higman.*

Chowla, S., Herstein, I. N., and Scott, W. R. The solutions of $x^d=1$ in symmetric groups. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 29–31 (1953).

Let $A_{n,d}$ be the number of solutions in the symmetric group S_n of the equation $x^d=1$, and form the generating function $\sum A_{n,d}x^n/n!=A_d(x)$. The authors prove that $A_d(x)=\exp\sum(x^k/k)$, where the last summation is restricted to positive integers k which divide d. This generalizes their previous result [Canadian J. Math. **3**, 328–334 (1951); these Rev. **13**, 10] for the number $A_{n,2}$, which is the sum of the degrees of the irreducible representations of S_n, and the result of Jacobsthal for the case when d is prime [Norske Vid. Selsk. Forh., Trondheim **21**, no. 12, 49–51 (1949); these Rev. **11**, 639]. *J. S. Frame.*

Mackey, George W. Symmetric and anti symmetric Kronecker squares and intertwining numbers of induced representations of finite groups. Amer. J. Math. **75**, 387–405 (1953).

Let $\mathcal{H}(U)$ be the space of a representation $U\colon x\to U_x$ of a finite group $\mathfrak{G}$ in a field of characteristic $\neq 2$. The space $\mathcal{H}(U\otimes U)$ of the Kronecker square $U\otimes U$ is the set of all linear transformations T from the conjugate $\overline{\mathcal{H}(U)}$ of $\mathcal{H}(U)$ to $\mathcal{H}(U)$, and $(U\otimes U)_x(T)=U_xTU_x^*$. The space $\mathcal{H}(U\otimes U)$ is decomposed into two subspaces consisting of T with $T=T^*$ and $T=-T^*$, respectively, and the representations of $\mathfrak{G}$ defined by the subspaces are called the symmetric and anti-symmetric Kronecker squares of U and denoted by $U\boxed{S}U$ and $U\boxed{A}U$, respectively. The intertwining number $\vartheta(\bar{U}, U)$ (=the dimensionality of the space of invariant elements in $\mathcal{H}(U\otimes U)$) is correspondingly decomposed into two numbers $\vartheta_S(\bar{U}, U)$ and $\vartheta_A(\bar{U}, U)$. Put $c(U)=\vartheta_S(\bar{U}, U)-\vartheta_A(\bar{U}, U)$. After a study of the symmetric and anti-symmetric squares and the invariants ϑ_S, ϑ_A, and c of a direct sum of representations, those of induced representations are investigated. Thus, if L is a representation of a subgroup G of $\mathfrak{G}$ and if U^L is that of $\mathfrak{G}$ induced by L, then $U^L\boxed{S}U^L$ (resp. $U^L\boxed{A}U^L$) is a direct sum of certain induced representations and the components are explicitly described. If U^L is monomial, so are the components. Corresponding decompositions of the numbers $\vartheta_S(\overline{U^L}, U^L)$, $\vartheta_A(\overline{U^L}, U^L)$ and $c(U^L)$ are given; the result for $c(U^I)$ includes a theorem of Frame [Bull. Amer. Math. Soc. **47**, 458–467 (1941); these Rev. **2**, 307]. A further study of the case $L=I$ gives a characterization of Wigner's [Amer. J. Math. **63**, 57–63 (1941); these Rev. **2**, 216] simply reducible groups in terms of double cosets in the triple direct product $\mathfrak{G}_3=\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}$. Let, in connection with Wigner's criterion, $v(x)$ be the order of the centrizer of x and $\zeta(x)$ the number of square roots of x. Then $\sum v(x)^n$ (resp. $\sum\zeta(x)^{n+1}$) is equal to $(\mathfrak{G}:1)$ times the number of double cosets (resp. self-inverse double cosets) in $\mathfrak{G}_n$ with respect to the diagonal. Thus $\sum\zeta(x)^{n+1}\leqq\sum v(x)^n$, and the conditions for equalities are studied; the case $n=2$ is Wigner's. A subsequent paper is promised in which the results will be extended to infinite groups. *T. Nakayama* (Nagoya).

Murai, Yasuhisa. On the group of transformations in six-dimensional space. Progress Theoret. Physics **9**, 147–168 (1953).

The author determines the continuous irreducible unitary representations of the (universal covering group of the component of the identity of the) group of all real linear transformations leaving invariant the quadratic form $x_1^2+x_2^2+x_3^2+x_4^2-x_5^2-x_6^2$. These are obtained in their infinitesimal form, along lines used by L. H. Thomas [Ann. of Math. (2) **42**, 113–126 (1941); these Rev. **2**, 216], and V. Bargmann [ibid. **48**, 568–640 (1947); these Rev. **9**, 133]. The representations are characterized by the numbers they associate with three elements of the enveloping algebra of the Lie algebra of the group, of degrees 2, 3, and 4. Arguments indicating possible physical significance for the present group have been given by the reviewer [Duke Math. J. **18**, 221–265 (1951); these Rev. **13**, 534] and others.
I. E. Segal (Chicago, Ill.).

Tits, J. Le plan projectif des octaves et les groupes de Lie exceptionnels. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 309–329 (1953).

It is known that the projective plane P where the coordinates are Cayley numbers is intimately connected with the exceptional Lie groups F_4 and E_6 [cf. H. Freudenthal, Oktaven, Ausnahmegruppen und Oktavengeometrie, Math. Inst. Rijksuniv., Utrecht, 1951; these Rev. **13**, 433; A. Borel, C. R. Acad. Sci. Paris **230**, 1378–1380 (1950); these Rev. **11**, 640]. The author reestablishes these results on the base of a direct study of the plane P. This plane is defined to be the set of pairs (x, y) of Cayley numbers, together with suitable points at infinity. Its straight lines have structures of 8-dimensional conformal spaces (spheres). Its

group of collineations is shown to be doubly transitive, simple, of dimension 78; from this the author deduces that it is a real form of E_6. Although the result is correct, the proof is insufficient, since the orthogonal group in 13 variables and the symplectic group in 12 variables are also of dimension 78. The author then studies the polarities of the plane P; they are of two kinds according as the set of united point is a real or imaginary hermitian conic. The group which leaves a polarity invariant is a real form of F_4; the compact form corresponds to the case of a polarity with no real united point. The author announces that the consideration of all hermitian conics in P leads to a geometric interpretation of the group E_7. *C. Chevalley* (Nagoya).

Freudenthal, Hans. Sur le groupe exceptionnel E_7. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 81–89 (1953).

E. Cartan discovered that the exceptional group E_7 admits a simple representation of degree 56; he indicated without proof that the image of E_7 under this representation is the group of all automorphisms which leave invariant both a nondegenerate alternating bilinear form and a certain quartic form J on the representation space [cf. E. Cartan, Thèse, Paris, 1894]. The author points out that the expression given by Cartan for J is incorrect; the same is true of his statement to the effect that the equations obtained by setting equal to 0 all partial derivatives of the second order of J define a variety of dimension 28. The author gives a correct expression for J. The space of the representation is defined to be the space of all pairs (X, Y) of skew-symmetric matrices of degree 8, and

$$J = \mathrm{Pf}\,(X) + \mathrm{Pf}\,(Y) - \tfrac{1}{4}\,\mathrm{Tr}\,(XY)^2 + \tfrac{1}{16}(\mathrm{Tr}\,XY)^2,$$

where Pf is the Pfaffian and Tr the trace. The author gives explicitly the infinitesimal transformations of E_7 (in its representation of degree 56) and shows that E_7 is the largest symplectic group which leaves J invariant. Moreover, he gives the equations of a 28-dimensional variety M which is invariant by E_7 and he constructs quantities which, under the operations of E_7, transform according to the adjoint representation of this group. He expresses the form J explicitly in terms of these quantities. The reviewer wishes to point out the following facts, which he has established himself: a) the group E_7 is of index 2 in the group of all transformations (not necessarily symplectic) which leave J invariant; b) M may be defined as the locus of singularities of the hypersurface $J=0$. *C. Chevalley* (Nagoya).

Freudenthal, Hans. Sur le groupe exceptionnel E_8. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 95–98 (1953).

The author gives a new representation of the infinitesimal transformations of the group E_8. Let M be a 9-dimensional vector space, E the space of endomorphisms of trace 0 of M, T the space of trivectors of M, and T^* the space of tricovectors; the Lie algebra of E_8 is taken to be the direct sum space $E+T+T^*$; the bracket operation is given in a very condensed form which exhibits a large degree of symmetry. *C. Chevalley* (Nagoya).

Freudenthal, Hans. Sur des invariants caractéristiques des groupes semi-simples. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 90–94 (1953).

Let g be a semi-simple Lie algebra over the field of complex numbers, and let $B(X, Y)$ be the bilinear form of Cartan on $g\times g$: $B(X, Y) = \mathrm{Tr}\,(\mathrm{ad}\,X)(\mathrm{ad}\,Y)$, where ad X is the image of X under the adjoint representation. Set

$$F(X, Y, Z) = B([X, Y], Z)$$

and

$$G(X, Y) = B([X, Y], [X, Y]).$$

Then the forms F and G are invariant with respect to the adjoint group of g. The author proves the following theorems. Assuming that g has no simple ideal of dimension 3, its adjoint group is the connected component of the identity in the groups of all operations which leave F invariant, and the same is true for G instead of F. Let f be an automorphism of the vector space g which leaves F (or G) invariant, and let h be a Cartan subalgebra of g. In order to prove that f is in the adjoint group (if it is sufficiently near the identity), one establishes first that it is sufficient to consider the case where f preserves h. For each root $\alpha\neq 0$, let E_α and $E_{-\alpha}$ be elements belonging to α and to $-\alpha$; then one proves by making use of the postulated invariance of F (or G) that, provided f is sufficiently near the identity, f maps into itself each one of the spaces $(h, E_\alpha, E_{-\alpha})$. This allows one to reduce the problem to the case where g is simple; actually, almost the whole burden of the proof consists in establishing the statement in the case where g is of type A_2. The conclusion on the top line of p. 93 does not seem to be warranted by the preceding formulas; but this does not matter, for the crucial equality $q_\alpha = 0$ is established on the middle of p. 93 without making use of this disputable conclusion. *C. Chevalley* (Nagoya).

Borel, A., et Serre, J.-P. Sur certains sous-groupes des groupes de Lie compacts. Comment. Math. Helv. **27**, 128–139 (1953).

The authors consider a topological group H which has a sequence of normal subgroups $e=H_0\subset H_1\subset\cdots\subset H_k=H$ such that every factor group H_i/H_{i-1} is either a finite cyclic group or a one-dimensional torus, and they prove that if such a group H is a subgroup of a compact Lie group G, it is contained in the normalizer N of a maximal torus T in G. It follows, in particular, that every abelian subgroup of G is contained in some N. On the other hand, the authors define the p-rank (p prime) of G as the largest integer h such that G contains the direct product of h cyclic groups of order p. It is clear that the p-rank of G is at least equal to the rank of G (i.e., the dimension of T), but, since the Weyl group N/T of G is finite, it follows from the above result that the p-rank of G is always finite. The authors then prove, using the method of spectral sequences of fibre bundles, that if the p-rank of a connected compact Lie group G is greater than the rank of G, G has p-torsion, and they show, in particular, that the exceptional Lie groups G_2, F_4, and E_8 have 2-torsions. *K. Iwasawa* (Cambridge, Mass.).

Yamabe, Hidehiko. On the conjecture of Iwasawa and Gleason. Ann. of Math. (2) **58**, 48–54 (1953).

This short paper states and proves the following theorem: a locally compact, connected group with a neighborhood U of the identity e which does not contain any non-trivial normal subgroup has a neighborhood V of e which does not contain any non-trivial group. The proof is accomplished through the more interesting lemma that given a neighborhood U of e there is a neighborhood V such that the set of elements of V each of which has all of its powers in V generate together a subgroup of U. Actually, the arguments of the paper establish a little more than is claimed: they show how to find in V a normal subgroup N of the given group G such that G/N does not have "small subgroups".

This is a generalization of results proved by Montgomery and the reviewer [Ann. of Math. (2) **56**, 213–241 (1952); these Rev. **14**, 135]. In a succeeding paper the author will show that a locally compact connected group without small subgroups is a Lie group. This paper and its successor eliminate all restriction to finite dimensionality in the results previously proved by Gleason [ibid. **56**, 193–212 (1952); these Rev. **14**, 135] and by Montgomery and Zippin [loc. cit.]. The methods follow closely those of Gleason [loc. cit.] but are sufficiently more powerful to achieve the greater generality with a simplification of proofs. There is a misprint in the paper in the parenthetic remark on page 50 which defines the symbol $x\psi(t)$: the equation should read $x\psi(t)=\psi(x^{-1}t)$.

L. Zippin (Flushing, N. Y.).

Newburgh, J. D. Metrization of finite dimensional groups. Duke Math. J. **20**, 287–293 (1953).

This paper proves in several cases that a connected locally compact finite-dimensional group is separable metric, that is, for the cases where any one of the following additional hypotheses is made the group is (1) compact, (2) locally connected, (3) a generalized Lie group or L-group, (4) one-dimensional. Since this paper was written, it has been shown by Yamabe [in the paper reviewed directly above and the sequel to appear] that every locally compact group is a generalized Lie group so that (3) is now known to be the general case, as Newburgh says he expected. The proofs are based on considerations about inverse limits and covering groups of Lie groups. *D. Montgomery.*

Riss, J. Eléments de calcul différentiel et théorie des distributions sur les groupes abéliens localement compacts. Acta Math. **89**, 45–105 (1953).

The distributions of Laurent Schwartz are continuous linear functionals on a certain suitably topologized vector space of infinitely differentiable functions on E_n. The success with which much of harmonic analysis has been carried over from E_1 and E_n to general locally compact Abelian groups suggests trying to carry out a corresponding generalization of the theory of distributions.

Schwartz's definition generalizes in a straightforward manner as soon as one has a notion of infinitely differentiable function and this is provided by differentiation with respect to one-parameter subgroups. The second half of the paper under review is devoted to exploring some of the properties of the distributions so defined and the first half to necessary background material on differentiation. Of course not every locally compact Abelian group admits nontrivial one-parameter subgroups, but this fact need not be taken explicitly into account. Distributions then degenerate into measures but the theorems proved are still true as stated.

One of the chief differences between the generalized theory introduced here and the classical one of Schwartz lies in the possibility of infinitely many linearly independent directions of differentiation; more precisely, the set of one-parameter subgroups forms a vector space in a natural way and this vector space may be infinite-dimensional. A principal result of part I shows that this circumstance is not as serious as it might be. If f is continuously differentiable on G and the derivative in each direction is a bounded function, then there exists a compact subgroup H of G such that f is constant on the H cosets and such that G/H has only finitely many linearly independent one-parameter subgroups. Another principal result of part I is the existence theorem on infinitely differentiable functions. Given any open subset O of G there exists a non-negative not identically zero such function whose support is in O.

The first half of part II is concerned mainly with the properties of $D(G)$ and $D'(G)$ as topological vector spaces where $D(G)$ is the space of infinitely differentiable functions with compact support and $D'(G)$ is the space of distributions. The topology of $D(G)$ is defined in three equivalent ways and it is shown to be complete. Four distinct topologies are introduced into $D'(G)$ three of which coalesce when G is E_n. Convolution of distributions is also discussed.

In the final half of part II Fourier transforms of distributions and distributions with point support are considered. As in the case considered by Schwartz the definition of distribution must be altered slightly when considering Fourier transforms. The so-called S distributions are continuous linear functionals on a somewhat different space of infinitely differentiable functions. The main results in this part of the paper give the structure of ordinary and S distributions with point support. *G. W. Mackey* (Cambridge, Mass.).

NUMBER THEORY

Thébault, Victor. Curiosités arithmétiques. Mathesis **62**, 120–129 (1953).

Griselle, Thomas. Proof of Fermat's last theorem for $n=2(8a+1)$. Math. Mag. **26**, 263 (1953).

Utz, W. R. A note on the Scholz-Brauer problem in addition chains. Proc. Amer. Math. Soc. **4**, 462–463 (1953).

An addition chain for the positive integer n is defined as a sequence of integers $a_0=1<a_1<a_2<\cdots<a_r=n$ where for each $i>0$, $a_i=a_j+a_k$ for some $j<i$, $k<i$, $j=k$ permitted. $l(n)$ denotes the smallest possible value of n. Scholz conjectured that $l(2^q-1)\leqq q+l(q)-1$. The author proves this if $q=2^s(2^n+1)$. *P. Erdös* (South Bend, Ind.).

Salié, Hans. Über abundante Zahlen. Math. Nachr. **9**, 217–220 (1953).

An integer m is called λ-abundant if $S(m)/m\geqq\lambda$, where $S(m)$ denotes the sum of divisors of m. Sharpening a previous result of Grün the author proves that the least prime factor of every λ-abundant number having n prime factors is less than $c(n\log n)^{1/\lambda}$. The author also discusses highly composite and superabundant numbers [see the paper of Alaoglu and the reviewer, Bull. Amer. Math. Soc. **50**, 881–882 (1944); these Rev. **6**, 117]. *P. Erdös.*

Kustaanheimo, Paul. On the fundamental prime of a finite world. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **129**, 7 pp. (1952).

The author previously noted [Soc. Sci. Fenn. Comment. Phys.-Math. **15**, no. 19 (1950); these Rev. **12**, 630] that for small residues modulo a prime p to have the order properties of rational numbers, it was necessary that -1 be a quadratic non-residue and that 2, 3, $\cdots$ up to some small q be quadratic residues. If our world is a geometry over some finite field with p elements, the author asks what can be said in terms of q about this fundamental prime p. This is essentially the problem of the least quadratic non-residue for a prime p. *Marshall Hall* (Columbus, Ohio).

Borel, Emile. Sur les intervalles séparant deux nombres premiers consécutifs. C. R. Acad. Sci. Paris 236, 1713 (1953).

Dans un livre récent sur les nombres premiers [Univ. de France, 1953; ces Rev. 14, 620], j'ai indiqué que les plus grands intervalles séparant deux nombres premiers consécutifs paraissaient inférieurs aux résultats déduits du calcul des probabilités. Ce fait est confirmé par une statistique faite sur une table récente de nombres premiers publiée dans Palamà et Poletti, Boll. Un. Mat. Ital. (3) 8, 52–58 (1953) [ces Rev. 14, 846]. *Author's summary.*

Misra, B. On Bose numbers. Amer. Math. Monthly 60, 319–322 (1953).

Let m/n be a rational number whose decimal expansion is desired. In case n is prime to 10 one can solve the linear diophantine equation $10b-ne=1$ for b (the "Bose number") and e. The author notes that division by n may be replaced by division by the smaller number b. The method is a variation of a method of R. C. Das [same Monthly 56, 87–89 (1949)]. *D. H. Lehmer* (Berkeley, Calif.).

Kontorovič, P. G., and Mil'man, D. I. On a method of N. I. Lobačevskiĭ for finding integer solutions of linear homogeneous equations with integer coefficients. Uspehi Matem. Nauk (N.S.) 8, no. 1(53), 145–149 (1953). (Russian)

The general integer solution of $a_1x_1+\cdots+a_nx_n=0$, where $a_1, \cdots, a_n$ are integers, as given in Skolem's Diophantische Gleichungen [Springer, Berlin, 1938] and ascribed to Betti (1862; cf. Skolem, op. cit., p. 4, 121) is to be found already in Lobačevskiĭ's course "Algebra" [Kazan Univ., 1834], although this point has not been made before. It is shown that his method also gives the most general solution of the independent equations $\sum_{i=1}^{n}a_{ji}x_i=0$ $(1\leqq j\leqq m<n)$ in the form

$$x_i=d^{-1}\sum(-1)^{i+\gamma}A_{\alpha_1,\alpha_2,\cdots,\alpha_{n-m-1},i}t_{\alpha_1,\cdots,\alpha_{n-m-1}}$$

where the t's are independent integer parameters, $A_{\alpha_1,\cdots,\alpha_{n-m}}$ is the determinant obtained by deleting columns $\alpha_1, \cdots, \alpha_{n-m}$ of the matrix of the equations, d is the greatest common divisor of the A's, and γ is the number of inversions in $\alpha_1, \cdots, \alpha_{n-m-1}, i$. *J. W. S. Cassels.*

Gloden, A. Une méthode de résolution de l'équation diophantienne $2x^2+1=ay^2$, $(a=2b^2+1)$. Bull. Soc. Roy. Sci. Liège 22, 195–196 (1953).

Mills, W. H. A system of quadratic Diophantine equations. Pacific J. Math. 3, 209–220 (1953).

The system is $x|y^2+ay+1$; $y|x^2+ax+1$, where a is a fixed integer. If x, y satisfy these relations, then so do y, z, where $xz=y^2+ay+1$. In this way solutions occur in chains $\cdots, u_{-1}, u_0, u_1, u_2, \cdots$ with $u_{n+1}u_{n-1}=u_n^2+au_n+1$. Each such chain satisfies a linear relation $u_{n+1}-bu_n+u_{n-1}+a=0$, where b is a constant characteristic of the chain: and the number of chains is finite for $a\neq\pm2$. Possible b are listed for $0\leqq a\leqq 10$. *J. W. S. Cassels* (Cambridge, England).

Xeroudakes, Georgios F. On the equation

$$x^4+mx^2y^2+ny^4=z^2.$$

Bull. Soc. Math. Grèce 27, 85–91 (1953). (Greek. English summary)

Let p, q, m be given integers, $q\neq1$, $(p,q)=1$, and assume the congruence $z^2\equiv p^2(p^2+mq^2) \pmod{q^4}$ has a solution z_0. The author proves that $x=p$, $y=q$ is a solution of the equation in the title if and only if n has the form

$$n=q^4X^2+2z_0X+[z_0^2-p^2(p^2+mq^2)/q^4],$$

where X is an integer. *T. Apostol* (Pasadena, Calif.).

Moser, Leo. On the diophantine equation

$$1^n+2^n+3^n+\cdots+(m-1)^n=m^n.$$

Scripta Math. 19, 84–88 (1953).

By employing certain congruence properties of sums of nth powers, the author proves that the equation of the title has no solutions in positive integers m and $n>1$ with $m\leqq10^{10^6}$. P. Erdös has conjectured that there are no solutions. *I. Niven* (Eugene, Ore.).

Georgiev, G. On the solution in rational numbers of the indeterminate equation

$$\sum_{k=1}^{n}A_kx_k^{m_k}=0.$$

Uspehi Matem. Nauk (N.S.) 8, no. 1(53), 127–130 (1953). (Russian)

If the integer indices m_k of the title equation can be divided into two nonvacuous sets, every element of the one set being coprime to every element of the other, then there is a reversible transformation of the type $x_i=\prod_{r=1}^{n}X_r^{\lambda_{ri}}$, where the λ_{ri} are integers and det $(\lambda_{ri})=\pm1$ such that in the transformed equation X_n occurs only to the powers λ, $\lambda+1$ (λ, some integer). Hence, if $X_1, \cdots, X_{n-1}$ are given, then X_n is determined by a linear equation, and so there is a parametric solution. This improves results of Chr. Karanikolov [Thesis, Sofia, 1942]. *J. W. S. Cassels.*

Georgiev, G. On the indeterminate equation

$$\sum_{k=1}^{m}A_k\prod_{i=1}^{n}x_i^{a_{ki}}=A_0.$$

Uspehi Matem. Nauk (N.S.) 8, no. 1(53), 131–134 (1953). (Russian)

A typical result is: if $m=n-1$, then a necessary and sufficient condition for the title equation to be reducible to $\sum A_kX_k=A_0$ by a transformation of the type considered in the preceding review is that the $m\times n$ minors of the matrix (a_{ik}) be relatively prime. *J. W. S. Cassels.*

Mikolás, Miklós. Über summatorische Funktionen von Möbiusschem Charakter. C. R. Acad. Bulgare Sci. 4 (1951), no. 2–3, 9–12 (1953). (Russian summary)

It is well known that the Möbius μ function satisfies $M(x)=\sum_{k=1}^{x}\mu(k)=o(x)$, and $M(x)=o(x^{\frac12+\epsilon})$ is equivalent to the Riemann hypothesis. The author says that $g(n)$ has Möbius characteristics if $\sum_{k=1}^{x}g(k)=o(x)$ and $\sum_{k=1}^{x}g(k)=o(x^{\frac12+\epsilon})$ implies the Riemann hypothesis. In several recent papers the author investigated this property [these Rev. 11, 645; 13, 627]. In the present note he proves that if $h(n)=\sum_{d|n}g(d)$ is strongly multiplicative and $h(n)=O(n^{-\frac12})$, then $g(n)$ has Möbius characteristics. *P. Erdös* (South Bend, Ind.).

Perron, Oskar. Über eine Formel von Ramanujan. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1952, 197–213 (1953).

The formula referred to in the title gives the value of a certain continued fraction and may be written

$$x+\frac{1^2-n^2}{x}\,\frac{}{+}\,\frac{2^2-m^2}{x}\,\frac{}{+}\,\frac{3^2-n^2}{x}\,\frac{}{+}\,\frac{4^2-m^2}{x}\,\frac{}{+\cdots}=m\frac{1+P(x)}{1-P(x)}$$

where

$$P(x)=\prod_{\delta^2=\epsilon^2=1}\left\{\frac{\Gamma\{(1+x+\delta m+\delta\epsilon n)/4\}}{\Gamma\{(3+x+\delta m+\delta\epsilon n)/4\}}\right\}^s.$$

This formula was discovered by Ramanujan who gave no proof however. A proof was given in 1930 by C. T. Preece [J. London Math. Soc. 6, 22–32 (1931)] using an argument involving asymptotic expansions. The author criticizes this proof and supplies an elementary proof based on the transformation of Bauer and Muir. This is elaborated sufficiently to prove the formula for all complex m, n and for $R(x)>0$. For $m=n$ we obtain a special case discovered by Stieltjes in 1891. *D. H. Lehmer* (Berkeley, Calif.).

Anderson, Douglas R., and Apostol, T. M. The evaluation of Ramanujan's sum and generalizations. Duke Math. J. 20, 211–216 (1953).

If $f(n)$ and $h(n)$ are multiplicative, then the sum

$$S(n;k)=\sum_{d\mid(n,k)}f(d)h(k/d)$$

is multiplicative in both variables. The authors consider in some detail the case that $f(n)$ is completely multiplicative, and $h(n)=0$ for all n which are not squarefree. In the general case they remark that $S(n,k)$ is a periodic function of k with period n. It follows that there is a finite Fourier series

$$S(n;k)=\sum_{m \bmod k}a_k(m)\exp(2\pi i nm/k),$$

where $a_k(m)=k^{-1}\sum_{d\mid(m,k)}dh(d)f(k/d)$. A number of special cases are indicated; the case $f(n)=n$, $h(n)=\mu(n)$ gives the formula for the Ramanujan sum. The authors also consider the series

$$\sum_{n=1}^{\infty}S(n;k)n^{-s}=\zeta(s)\sum_{d\mid k}f(d)h(k/d)d^{-s}.$$

N. G. de Bruijn (Amsterdam).

Newman, Morris. The coefficients of certain infinite products. Proc. Amer. Math. Soc. 4, 435–439 (1953).

Put $\prod_{n=1}^{\infty}(1-x^n)^r=\sum_{n=0}^{\infty}p_r(n)x^n$. For $r=1$, 3 we have the familiar identities of Euler and Jacobi, respectively, but as the author remarks, for no other values of r are the coefficients $p_r(n)$ known "explicitly". In the present paper various recursion formulas and identities are set up. The identity

(*) $\sum p_r(np+\delta)=p_r(\delta)\prod(1-x^n)^r-p^{r/2-1}x^\delta\prod(1-x^{np})^r,$

where $1\leqq r\leqq 24$, $r(p-1)\equiv 0 \pmod{24}$, $\delta=r(p-1)/24$, is quoted from a previous paper [Trans. Amer. Math. Soc. 73, 313–320 (1952); these Rev. 14, 250]. It is then proved that

(**) $$\sum p_r(np+\Delta)=(-p)^{r/2-1}\prod(1-x^{np})^r,$$

where $r=2$, 4 or 6, $r(p+1)\equiv 0$ (24), $\Delta=r(p^2-1)/24$. By means of (**) results like

$$p_r(r(p^{2t}-1)/24)=(-p)^{(r/2-1)t}$$

are proved; a like result derived from (*) implies that every integer occurs infinitely often as the modulus of the coefficients $p_2(n)$. *L. Carlitz* (Durham, N. C.).

Tamagawa, Tsuneo. On the functional equation of the generalized L-function. J. Fac. Sci. Univ. Tokyo. Sect. I. 6, 421–428 (1953).

In a recent paper A. Weil [J. Math. Soc. Japan 3, 1–35 (1951); these Rev. 13, 439] introduced a generalization of Artin's nonabelian L-series; for a complete description of this generalization see the last three paragraphs of the review cited. The present paper contains a derivation, by methods similar to Artin's, of the functional equation of these new L-functions. *G. Whaples.*

Cugiani, Marco. Sulle funzioni simmetriche delle radici dell'unità (mod p^α). Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 15(84), 651–662 (1951).

For an odd prime p, $\alpha\geqq 1$, $n=rp^\alpha$, $\rho=\lambda p^\pi$ where $0\leqq\pi\leqq\alpha-1$ and λ is an even divisor of $p-1$, define W_k, a function of k, p, π, α, λ, by the relation

$$\prod_i(x+t_i)=\sum_{k=0}^{\rho r}W_kx^{\rho r-k}$$

where t_i ranges over $0<t_i<n$ such that $t_i^\rho\equiv 1 \pmod{p^\alpha}$. For $k>\rho r$, define $W_k=0$. Let max (p,a) denote the exponent of the highest power of p which divides a, and define $\beta=\beta(k)=1+\max(p,W_k)$ and $m=rp^\pi$. It is proved that $\beta-1\geqq\max(p,\binom{m}{s+1})$ for $k=s\lambda+h$, $1<h<\lambda$; that

$$W_k\equiv(-1)^s\binom{m}{s}\pmod{p^\beta}$$

for $k=s\lambda$; that $W_k\equiv\frac{1}{2}(-1)^sn\lambda(s+1)\binom{m}{s+1}\pmod{p^\beta}$ for $k=s\lambda+1$. Results are also obtained for the case $p=2$. This generalizes earlier work of the author [same Rend. (3) 14(83), 529–543 (1950); these Rev. 14, 20]. *I. Niven.*

Carlitz, L. Some theorems on the Schur derivative. Pacific J. Math. 3, 321–332 (1953).

Given the sequence $\{a_m\}$ and $p\neq 0$, the Schur derivative a_m' is defined by $a_m'=\Delta a_m=(a_{m+1}-a_m)/p^{m+1}$, and the higher derivatives are defined by means of $a_m^{(r)}=\Delta^r a_m=\Delta(a_m^{(r-1)})$, $a_m^{(0)}=a_m$. Using p-adic methods, Zorn [Ann. of Math. (2) 38, 451–464 (1937)] proved that if p is a prime and if we define $X_m=(x^{p^m}-1)/p^{m+1}$ for $x\equiv 1 \pmod p$, then

$$\Delta^rX_m\equiv\frac{(p-1)(p^2-1)\cdots(p^r-1)}{(r+1)!}X_m^{r+1}\pmod{p^m}$$

provided $r<p$; for $r<p-2$ the congruence holds $\pmod{p^{m+1}}$. An easy consequence is the theorem of Schur [S.-B. Preuss. Akad. Wiss. 1933, 145–151] which states that if a is an integer, $p\nmid a$ and $a_m=a^{p^m}$, then the derivatives $\Delta^ra^{p^m}$ are integral for $1\leqq r\leqq p-1$; moreover, if $a_0'\equiv 0 \pmod p$, then all the derivatives $\Delta^ra^{p^m}$ are integral, while if $a_0'\not\equiv 0 \pmod p$, then the numbers $\Delta^pa^{p^m}$ have the denominator p.

In the present paper the author gives a simple elementary proof of Zorn's congruences and also proves a number of related results. He next extends Schur's and Zorn's theorems to algebraic numbers. Then he considers a generalization of another kind suggested by the arithmetic function $F(a,m)=\sum_{de=m}\mu(d)a^e$. Finally, he gives some applications of Schur's theorem to the Euler and Bernoulli polynomials and numbers; the results are analogous to Kummer's congruences [cf. N. E. Nörlund, Vorlesungen über Differenzenrechnung, Springer, Berlin, 1924, Ch. 12]. In particular $\Delta^rE_{k+p^m}$ is integral $\pmod p$ for $p>2$, $r<p$, $r\leqq m$; also $\Delta^r(B_{k+p^m}/(k+p^m))$ is integral $\pmod p$ for $p-1\nmid k+1$, $r<p$, $r\leqq m$. Here E_k and B_k denote the Euler and Bernoulli numbers in the notation of Nörlund. *A. L. Whiteman.*

Carlitz, L. A theorem on congruences. J. Indian Math. Soc. (N.S.) 17, 43–45 (1953).

Let $f(x)$ denote a polynomial with rational integral coefficients and discriminant D. In this note the author proves the theorem that if $f(x)\equiv 0 \pmod{p^\alpha}$ is solvable for $\alpha=d+1$, where p^d is the highest power of the prime p dividing D,

then the congruence is solvable for all a. While this theorem is contained in a more general theorem of Hensel [cf. K. Hensel, Theorie der algebraischen Zahlen, Teubner, Leipzig-Berlin, 1908, p. 68], the present proof is simple and direct.
A. L. Whiteman (Princeton, N. J.).

Carlitz, L. A note on partitions in $GF[q, x]$. Proc. Amer. Math. Soc. 4, 464–469 (1953).

For given $A \in GF(q, x)$, $A \neq 0$, let $p(A)$ denote the number of solutions U_i of

$$(*) \quad A = U_1 + U_2 + \cdots \quad (\deg A = \deg U_1 > \deg U_2 > \cdots);$$

let $r(A)$ denote the number of solutions of (*) such that $\deg U_i = m - i + 1$ $(i = 1, 2, \cdots, m+1)$, where $m = \deg A$; let $p_k(A)$ denote the number of solutions of $A = U_1 + \cdots + U_k$ $(\deg U_1 > \deg U_2 > \cdots)$. The author proves the following results:

$$(1) \qquad p(A) = \prod_{i=0}^{m-1} (1 + (q-1)q^i);$$

$$(2) \qquad r(A) = (q-1)^m q^{m(m-1)/2};$$

$$(3) \qquad p_k(A) = q^{(k-1)(k-2)/2}(q-1)^{k-1}\left[{m \atop k-1}\right] \quad (k \geqq 1),$$

where

$$\left[{m \atop k}\right] = (q^m - 1) \cdots (q-1)/(q-1) \cdots (q^k - 1) = \left[{m \atop m-k}\right], \quad \left[{m \atop 0}\right] = 1.$$

Analogous results are obtained for the corresponding functions $p'(A)$, $r'(A)$, $p_k'(A)$ in which the solutions involved are subject to the additional restriction that all the polynomials are primary. A special quadratic partition problem similar to that concerning $r(A)$ and $r'(A)$ is also considered. The principal tool employed is the method of generating functions. *A. L. Whiteman* (Princeton, N. J.).

Dinghas, Alexandre. Sur un théorème de Schur concernant les racines d'une classe des équations algébriques. Norske Vid. Selsk. Forh., Trondheim 25 (1952), 17–20 (1953).

Let S be the set of all polynomial equations with leading coefficient 1, integral coefficients, and all roots positive, simple, and distinct. I. Schur [Math. Z. 1, 377–402 (1918)] proved that if $\gamma < e^{1/2}$, there are only a finite number of polynomials $f(x) \in S$ with $(x_1 + x_2 + \cdots + x_n)/n < \gamma$ where $x_1, \cdots, x_n$ are the roots. C. L. Siegel somewhat improved the bound $e^{1/2}$ [Ann. of Math. (2) 46, 302–312 (1945); these Rev. 6, 257]. This paper sketches a proof of the following generalization of these results: There is a number r_0 with $1/2 < r_0 \leqq 2/3$ and, for all $r > r_0$, a positive $\delta(r)$, such that for every $r > r_0$, $0 < \delta < \delta(r)$, the number of equations in S with $x_1^r + x_2^r + \cdots + x_n^r \leqq (1+\delta)n$ is finite. *G. Whaples.*

Flanders, Harley. Generalization of a theorem of Ankeny and Rogers. Ann. of Math. (2) 57, 392–400 (1953).

E. Trost [Nieuw Arch. Wiskunde 18, 58–61 (1934)] and, later and independently, N. C. Ankeny and C. A. Rogers [Ann. of Math. (2) 53, 541–550 (1951); these Rev. 12, 804; Trost's paper was unfortunately unknown to the reviewer when this review was written] proved the following result: if a rational integer is an n-power residue for "almost all" prime numbers p (i.e., for all, except some primes, forming a set of kroneckerian density 0), it is either nth power (in the rational field) or it is the product of an nth power by $2^{n/2}$ and 8 divides n.

The author generalizes this result to algebraic number fields of finite degree and to algebraic function fields of one variable with a finite coefficient field, also of finite degree. In the hypothesis of the statement, the prime numbers are replaced by the prime ideals of the field. In the case of algebraic numbers, if ν is the largest number such that $\rho_\nu = \cos 2^{-\nu-1}\pi$ belongs to the considered field, the number $2^{n/2}$ is replaced in the conclusion of the statement by $[\frac{1}{2}(1+\rho_\nu)]^{n/2}$ and the condition that 8 divides n is replaced by: $2^{\nu+1}$ divides n and the field does not contain $i = \sqrt{-1}$ nor $i\rho_{\nu+1}$. In the algebraic functions case, n is supposed prime to the characteristic of the residual field, and an element of the field is always an nth power in the field if it is an n-power residue for almost all ideals of this field.

The method is exactly the same as that of Ankeny and Rogers, with only slight technical complications.
M. Krasner (Paris).

Remak, Robert. Über Grössenbeziehungen zwischen Diskriminante und Regulator eines algebraischen Zahlkörpers. Compositio Math. 10, 245–285 (1952).

The author has previously found [J. Reine Angew. Math. 167, 360–378 (1932)] a lower bound $R_{\min}$ for the regulator R of the units of an algebraic number field which depends only on the number r of real conjugates and the number s of pairs of complex conjugates ($n = \text{degree} = r + 2s$; $k = \text{number}$ of independent units mod the roots of unity $= r + s - 1$). In this paper he studies the problem of whether an upper bound on R implies an upper bound on the disciminant D; that is, whether there is a theorem $|D| < f(|R|, n)$. He finds that this is so for all fields except those which have a unit defect. A field is said to have a unit defect if it contains a subfield which has the same number of independent units as the original field. The fields with unit defect are precisely those which are total-imaginary and are of degree 2 over some total-real subfield. For fields without unit defect, $\log D \leqq n \log n + (|R|/R_{\min}) \log 2(2n(n-1)/(k+1))g^{(k-1)/2}$ where $R_{\min}$ and k are defined above and

$$g = (k + 5 + (k^2 + 2k + 17)^{\frac{1}{2}})/2.$$

In certain cases this bound can be improved. The method is based on producing a unit which cannot lie in any subfield, by use of an unsymmetrical convex body. Naturally there are many complicated details. A unit defect is said to be strong when all the units of the field lie in some proper subfield, otherwise weak. It is proved that no relation $|D| \leqq f(|R|, n)$ can hold for fields with strong unit defect but it is stated that such a relation does hold for those with weak unit defect. *G. Whaples* (Bloomington, Ind.).

Nakayama, Tadasi. On the cohomology of algebraic number fields. Sûgaku 4, 129–137 (1952). (Japanese)

This is mostly an expository paper on the cohomology of algebraic number fields which has been developed recently by the author, Hochschild, Weil, and Artin and Tate. In particular, the author summarizes the main results and ideas of the following papers: Nakayama, Ann. of Math. (2) 55, 73–84 (1952) [these Rev. 13, 629]; Hochschild and Nakayama, ibid. 55, 348–366 (1952) [these Rev. 13, 916], and Nakayama, ibid. 57, 1–14 (1953) [these Rev. 14, 453].
Y. Kawada (Princeton, N. J.).

Moriya, Mikao. Theorie der Derivationen und Körperdifferenten. Math. J. Okayama Univ. 2, 111–148 (1953).

Y. Kawada, following an idea of A. Weil, has defined the different for rings of algebraic integers in terms of the notion of derivation [cf. A. Weil, Bull. Amer. Math. Soc. 49, 41 (1943); Y. Kawada, Ann. of Math. (2) 54, 302–314 (1951); these Rev. 13, 324]. The author generalizes this procedure

to the case of any ring in which the classical Dedekind ideal-theory holds (contrary to a usage which is becoming common, he calls these rings "Noether rings"). Let $\mathfrak{o}$ be such a ring and k its field of quotients. Let K/k be a separable finite algebraic extension, and $\mathfrak{O}$ the ring of elements of K which are integral over $\mathfrak{o}$. Let $\mathfrak{A}$ be an ideal $\neq\{0\}$ in $\mathfrak{O}$; then $\mathfrak{O}/\mathfrak{A}$ is an $\mathfrak{O}$-module. We denote by $\mathfrak{D}(\mathfrak{O};\mathfrak{o};\mathfrak{A})$ the set of derivations of $\mathfrak{O}$ into $\mathfrak{O}/\mathfrak{A}$ which map $\mathfrak{o}$ upon $\{0\}$; this is again an $\mathfrak{O}$-module. This module satisfies the chain condition and has therefore a Jordan-Hölder length $d(\mathfrak{A})$ (which the author calls its dimension). It is proved that $d(\mathfrak{A})$ remains bounded for all $\mathfrak{A}$; let d be the largest of the numbers $d(\mathfrak{A})$. In order for $d(\mathfrak{A})$ to equal d, it is necessary and sufficient that $\mathfrak{A}$ be a multiple of a certain ideal $\mathfrak{D}_0$, which the author calls the quasi-different of $\mathfrak{O}$. Let $\mathfrak{D}_0=\prod\mathfrak{P}_i^{e(i)}$ be the prime ideal decomposition of $\mathfrak{D}_0$, and let $d(i)$ be the length of $\mathfrak{D}(\mathfrak{O};\mathfrak{o};\mathfrak{P}_i^{e(i)})$. Then it is proved that $\mathfrak{D}=\prod\mathfrak{P}_i^{d(i)}$ is the Dedekind different of $\mathfrak{O}/\mathfrak{o}$. Moreover, $\mathfrak{D}$ is always a multiple of $\mathfrak{D}_0$; we certainly have $\mathfrak{D}=\mathfrak{D}_0$ if, for every prime ideal $\mathfrak{P}$ of $\mathfrak{O}$, $\mathfrak{O}/\mathfrak{P}$ is separable over $\mathfrak{o}/(\mathfrak{P}\cap\mathfrak{o})$, and, in this case, $\mathfrak{D}(\mathfrak{O};\mathfrak{o};\mathfrak{A})$ is cyclic for any ideal $\mathfrak{A}\neq\{0\}$. In the case where $\mathfrak{o}$ is a complete valuation ring, the following conditions are equivalent to each other: a) $\mathfrak{D}=\mathfrak{D}_0$; b) for any $\mathfrak{A}\neq\{0\}$, $\mathfrak{D}(\mathfrak{O};\mathfrak{o};\mathfrak{A})$ is cyclic; c) $\mathfrak{D}$ is the H.C.D. of the differents of the elements of $\mathfrak{O}$. *C. Chevalley* (Nagoya).

***Takagi, Teiji. Daisûteki Seisûron. Gaisetsu oyobi Ruitairon. [Algebraic number theory. Generalities and class field theory.]** Iwanami Shoten, Tokyo, 1948. vi+316+v pp. 300 yen.

This is a revised edition of a monograph published in 1934 as one of the "Iwanami Sûgaku Kôza" (Iwanami Mathematical Series). In this book the author develops algebraic number theory, starting from the very beginning of the ideal theory of Dedekind and including all of class field theory. Class field theory was established by the author [J. Coll. Sci. Imp. Univ. Tokyo **41**, no. 9 (1920)] and was complemented by Artin's general reciprocity theorem [Abh. Math. Sem. Univ. Hamburg **5**, 353–363 (1927)]. This book was (and perhaps still is) the only complete exposition of class field theory in book form (except Hasse's "Berichte" [Jber. Deutsch. Math. Verein. **35**, 1–55 (1926); **36**, 233–311 (1927); ibid. Ergänzungsband **6**, 1–204 (1930)] and Hasse's mimeographed lecture notes [Klassenkörpertheorie, Marburg, 1933]). In this new edition the author corrects misprints carefully and completes several parts, so that now this is almost a complete description of class field theory from the ideal-theoretic point of view. After the first edition Chevalley [Ann. of Math. (2) **41**, 394–418 (1940); these Rev. **2**, 38] developed class field theory on the basis of his idèle theory and he replaced the analytic considerations in the proof of the fundamental theorem by a simpler arithmetic method. Here the author does not make use of this new method, but one can understand through this book how class field theory was developed from the classical tradition since Gauss, Hilbert and Furtwängler.

This book is divided into two parts. The first part deals with the general algebraic theory of number fields and consists of eleven chapters. Starting from the usual description of algebraic numbers, algebraic integers, and number fields (Chap. I), the fundamental theorem of the ideal theory of Dedekind is proved (Chap. II). Here the "Inhalt" of a polynomial with integer coefficients is introduced following Kronecker's tradition. In this new edition Appendix III is added, where the fundamental theorem of ideal theory is considered again. Here the author describes the method of Kronecker [see Werke, Bd. II, Teubner, Leipzig, 1897] and compares the three methods of Kronecker, Dedekind and Hurwitz [see Kronecker, ibid., p. 237]. In Chap. III he considers the residue classes of integers modulo ideals, and especially the structure of residue class fields modulo prime ideals. In Chap. IV he introduces fractional ideals and proves the finiteness of the ideal class groups. Chap. V contains Minkowski's theorem on linear forms and its application to the discriminants of number fields. Chap. VI is devoted to the ideal theory of relative extensions. Chap. VII deals with differents and discriminants. The author gives first the definitions of "elements" $\mathfrak{e}^{(i)}$ and different $\mathfrak{D}_k=\mathfrak{e}^{(1)}\cdots\mathfrak{e}^{(n-1)}$ after Hilbert. He proves then that the relative different $\mathfrak{D}_{K/k}$ is generated by $\mathfrak{D}_{K/k}(\theta)$ for all integers θ of K, the chain theorem $\mathfrak{D}_K=\mathfrak{D}_{K/k}\cdot\mathfrak{D}_k$ and the discriminant-theorem of Dedekind. He proves the last theorem by using Hensel's theorem on forms along the lines of Kronecker. Chap. VIII is an exposition of Hilbert's theory on the decomposition and ramification of prime ideals of k in a Galois extension K/k. In this case the discriminant-theorem can be proved easily. This chapter contains also the theory of cyclotomic fields k including theorems on the decomposition of prime numbers in k and a theorem of Kronecker-Weber that every abelian extension of the rational field is a cyclotomic field. Chap. IX is devoted to the theory of units. Dirichlet's unit theorem and Herbrand-Artin's theorem of units for relative Galois extensions are proved. Chap. X is a short comment on the $\mathfrak{p}$-adic numbers, and after introducing exp and log functions the author determines the indices of nth power residues mod $\mathfrak{p}^n$ for sufficiently large n. In this book $\mathfrak{p}$-adic extensions are used only as a supplementary method. Later only in the computation of the indices of norm residues mod $\mathfrak{p}^n$ are $\mathfrak{p}$-adic methods used again.

In the second part of this book class field theory is developed mainly after the author's original method. Of course it includes Artin's general reciprocity theorem. The author adopts some simplified methods of Artin (mimeographed notes of lectures in Göttingen (1932)), Hasse [lecture notes cited above], and Chevalley and Herbrand [Chevalley, J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. **2**, 365–476 (1933)]. In Chap. XI multiplicative congruences (with signatures) of numbers in k are considered, and the notions of an ideal group $H_\mathfrak{m}$ (defined mod $\mathfrak{m}$) and its conductor $\mathfrak{f}$ are defined carefully. Chap. XII is devoted to analytic considerations. First Dedekind's ζ-function over an algebraic number field k is introduced, then the L-functions $L(s,\chi)$ for characters χ of the multiplicative group $A_\mathfrak{m}$ of ideals mod $H_\mathfrak{m}$. The following analytic property of L is stated without proof: $L(s,\chi)$ can be analytically prolonged (on the whole s-plane), and is an integral function if $\chi\neq\chi_1$ and meromorphic for $\chi=\chi_1$. The function $L(s,\chi_1)$ has a pole of order 1 at $s=1$. (Of course we need fewer properties of L for the following applications, as the author remarks.) Here the class field K/k is defined in the same manner as in the author's original paper. Namely, let M be a set of prime ideals in k and define its density by $\Delta(M)=\lim_{s\to1+0}(\sum_{\mathfrak{p}\in M}(1/N\mathfrak{p}^s))/\log(1/s-1)$. For general Galois extension K/k, the density of the set M of all the prime ideals of k which decompose completely in K/k is $1/n$ $(n=[K:k])$. On the other hand the density of $M_1=\{\mathfrak{p}_1\}$ of all the prime ideals in an ideal group $H_\mathfrak{m}$ is either $1/h$ or 0 $(h=[A_\mathfrak{m}:H_\mathfrak{m}])$. Especially for the ideal group $H_\mathfrak{m}(K/k)$ which is generated by $N_{K/k}\mathfrak{A}$ mod $\mathfrak{m}$ we

have the relation $M_1 \supset M$, which implies the so-called second inequality $h \leqq n$. The conductor $\mathfrak{f}$ of K/k is defined here as that of $H(K/k)$ for which h has the maximum value. Now the class field K/k is defined (somewhat artificially) as a Galois extension for which $h=n$ holds. Then K/k is a class field if and only if almost all (in the sense of density) prime ideals $\mathfrak{p}$ in $H\mathfrak{f}(K/k)$ decompose completely in K/k. The generalized Dirichlet theorem for arithmetic progressions and the inequality $L(1, \chi_i) \neq 0$ ($\chi_i \neq \chi_1$) for χ_i of A/H follow easily from these analytic considerations, if we assume all the propositions of class field theory. In Chap. XIII the fundamental theorem to the effect that every abelian extension K/k is a class field is proved. This proof takes about 20 pages. First for cyclic extensions the first inequality $h \geqq n$ is proved arithmetically by making use of Herbrand's device. Combining this with the second inequality we get $h=n$. As a by-product we get the principal genus theorem for cyclic extensions. From the cyclic case follows the equality $h=n$ for the abelian case by the combination theorem of class fields: if K_1/k, K_2/k are class fields over H_1, H_2, respectively, then $K_1 \cdot K_2$ is the class field over $H_1 \cap H_2$. Next come the ordering theorem and the uniqueness theorem of class fields, which follow immediately from the combination theorem. In Chap. XIV the decomposition theorem to the effect that the exponent f of a prime ideal $\mathfrak{p}$ mod $H\mathfrak{f}(K/k)$ is equal to the relative degree of $\mathfrak{p}$ in K/k, and the isomorphism theorem, to the effect that the Galois group of K/k is isomorphic to $A\mathfrak{f}/H\mathfrak{f}(K/k)$, are derived from Artin's general reciprocity theorem. The latter is proved by the direct method due to Chevalley. In Chap. XV the existence theorem of class fields is proved following Herbrand's method. For that purpose some necessary properties of Kummer extensions are considered. Next comes the conductor theorem. Here Hasse's explicit formula for the conductor is referred to without proof. Chap. XIV contains the density theorem of Tschebotareff [Math. Ann. **95**, 191–228 (1925)]. The last theorem of the second part is a theorem of Hasse and Scholz [Math. Z. **29**, 60–69 (1928)] that a class field K/k is always abelian.

There are 3 Appendices of about 50 pages. Appendix I contains the theory of quadratic fields. This includes all the results of a long chapter on quadratic fields in Hilbert's "Zahlbericht" [Jber. Deutsch. Math. Verein. **4**, 175–546 (1897)]. Here this is treated as an application of class field theory so that the author can develop the theory in 10 pages. Appendix II is concerned with the class numbers of cyclotomic fields. The first ten chapters and Appendices I, II cover almost all important results given in Hilbert's Zahlbericht, chapters I–IV. Chap. V of Hilbert's Zahlbericht was devoted to the theory of Kummer fields, which was at that time the highest part of algebraic number theory. But now it is outdated and class field theory has taken its place.

In a supplement the author gives a short history of and references to complex multiplications, the principal ideal theorem, and Fermat's last theorem. In the last line of this book the author expresses his hope of extending class field theory to the non-abelian case in the future, and also of simplifying the present proofs from a new standpoint. The explanations in this book are quite clear and rather compact. As a detailed exposition of the rational and quadratic fields the author has already published another book entitled "Lectures on elementary number theory" [Takagi, Shotô Seisûron Kôgi, published by Kyôritsu Shuppan, Tokyo, 1931, 496 pp.]. In this book the author has treated in full detail the algebraic number theory of quadratic fields from a higher standpoint, and has given an exposition of the background and significance of this theory. These two books have contributed greatly to the progress of number theory in Japan. The reviewer hopes that they will be translated into some European language. *Y. Kawada.*

Cassels, J. W. S. Yet another proof of Minkowski's theorem on the product of two inhomogeneous linear forms. Proc. Cambridge Philos. Soc. **49**, 365–366 (1953).

1. If $\xi=\alpha x+\beta y$, $\eta=\gamma x+\delta y$ be homogeneous linear forms in x, y with real coefficients and determinant $\alpha\delta-\beta\gamma=\Delta\neq 0$, then for any real constants p, q there are integers x, y such that (1) $|\xi+p||\eta+q| \leqq \frac{1}{4}|\Delta|$. 2. If $\alpha x+\beta y\neq 0$ for all integers $x, y \neq 0, 0$, then there is a solution of (1) with $|\xi+p| \leqq \epsilon$ for every positive ϵ. These theorems are deduced in a few lines from the following lemma. Let θ, ϕ, ψ, ω be four real numbers such that $|\theta\omega-\phi\psi| \leqq \frac{1}{2}|\Delta|$, $|\psi\omega| \leqq |\Delta|$, $\psi>0$. Then there is an integer x such that $\lambda=\theta+\psi x$, $\mu=\phi+\omega x$ satisfy the inequalities $|\lambda\mu| \leqq \frac{1}{4}|\Delta|$, $|\lambda| \leqq \psi$. Proof. By writing $\theta+r\psi$, $\phi+r\omega$ for θ, ϕ with a suitable integer r, we may suppose $-\psi \leqq \theta<0$ and by writing $-\phi$, $-\omega$, $-\mu$ for ϕ, ω, μ, if need be, we also may suppose $\phi \geqq 0$. Then by distinguishing the cases $\phi+\omega \leqq 0$ and $\phi+\omega \geqq 0$ it is easily verified that $x=0$ or $x=1$ will do. *J. F. Koksma.*

Voronoĭ, G. F. Notes on indeterminate quadratic forms. Ukrain. Mat. Žurnal **3**, 240–271 (1951). (Russian)
Voronoĭ, G. F. On indeterminate quadratic forms. Ukrain. Mat. Žurnal **3**, 272–278 (1951). (Russian)
Venkov, B. A. On the scientific diary of G. F. Voronoĭ. Ukrain. Mat. Žurnal **3**, 279–289 (1951). (Russian)

Voronoĭ, who died in 1908 at the age of 40, is best remembered by his classical work on positive definite quadratic forms [J. Reine Angew. Math. **133**, 97–178 (1908); **134**, 198–287 (1909)] which even today are not yet exhuasted in their implications. Mathematicians will be grateful to Professor Venkov for preparing for print notes on indefinite quadratic forms taken from a diary kept by Voronoĭ shortly before his death.

The longer extract deals mainly with the decomposition of an indefinite quadratic form $f(x_1, \cdots, x_n)$ of characteristic (μ, ν), $\mu+\nu=n$, into a sum $\phi(u_1, \cdots, u_\mu)+\psi(v_1, \cdots, v_\nu)$ of a positive definite form ϕ and a negative definite form ψ; here $u_1, \cdots, u_\mu, v_1, \cdots, v_\nu$ are independent linear forms in $x_1, \cdots, x_n$. Such decompositions are studied from the algebraic and geometric standpoint, and they are connected by means of Hermite's method of continuous variables with the reduction theory of positive definite forms. The ternary case ($\mu=2$, $\nu=1$) is considered in detail.

The second shorter extract from the diary consists of some inconclusive results on the example $f=x^2+y^2-3z^2-3t^2$. In the third paper, Venkov comments on the interest of these notes and places them in their historical position with regard to classical and modern work on quadratic forms.

K. Mahler (Manchester).

Oppenheim, A. One-sided inequalities for hermitian quadratic forms. Monatsh. Math. **57**, 1–5 (1953).

Let $\phi(x, y)=ax\bar{x}+\beta x\bar{y}+\bar{\beta}\bar{x}y+cy\bar{y}$ be a binary Hermitian form, and let $\Delta=ac-\beta\bar{\beta}$ be its determinant. Let x, y be integers in $k(\sqrt{m})$. The form is indefinite if $m>0$, $\Delta\neq 0$ and also if $m<0$, $\Delta<0$. The author proves that the inequality

$$0<\phi(x, y) \leqq C|\Delta|^{1/2}$$

is soluble in integers x, y of $k(\sqrt{m})$, where $C=|m|^{1/2}$ if $m\equiv 1 \pmod 4$ and $C=2|m|^{1/2}$ otherwise, m being squarefree. If certain forms are excluded, the inequality is soluble with C replaced by $2^{-1/2}C$, and if certain other forms are excluded C can be replaced by $\frac{2}{3}C$. It is of interest that the results obtained are more complete than those known for $|\phi(x, y)|$. The proof is based on results for quaternary quadratic forms of signature zero, in papers by the author which are not yet published. There are some misprints, e.g., a factor a is missing from the first term on the right of (3).

H. Davenport (London).

Oppenheim, A. Values of quadratic forms. I. Quart. J. Math., Oxford Ser. (2) **4**, 54–59 (1953).

The following theorems are proved: (I) If $f(x_1, x_2, \cdots, x_n)$ is an indefinite quadratic form, such that for every $\epsilon>0$ the inequalities $0<f<\epsilon$ are solvable in integers x_r, then, if $n\geqq 3$, the inequalities $0<-f<\epsilon$ are solvable in integers x_r for every $\epsilon>0$. The theorem does not hold in general for $n=2$, as shown by a counter-example. (II) If f is an indefinite form with non-zero determinant $\Delta(f)$, then to every positive value a taken by f corresponds a negative value $-b$ taken by f such that $b^{2n-2}\leqq A_n a^{n-2}|\Delta(f)|$, where the constant A_n depends only on n. Both a and b can be properly represented values. (III) If $P_1(f)$ denotes the lower bound of the positive values of a positive definite or indefinite form f, then $P_1^n(f)\leqq B_n|\Delta(f)|$, where B_n is a constant which depends only on n. Theorem (I) is a corollary of theorem (II). Theorems (II) and (III) are proved together by induction on n. A different proof of theorem (III), based on the adjoint form of f, is also given, establishing at the same time the inequality $B_n\leqq B_{n-1}^{n/(n-2)}$. For theorem (III) see also Blaney, J. London Math. Soc. **23**, 153–160 (1948); these Rev. **10**, 511.

E. Grosswald (Philadelphia, Pa.).

Oppenheim, A. Values of quadratic forms. II. Quart. J. Math., Oxford Ser. (2) **4**, 60–66 (1953).

It has been conjectured that if f is an incommensurable quadratic form in 5 or more variables, of non-vanishing determinant, then $0<|f(x_1, x_2, \cdots, x_n)|<\epsilon$ is solvable in integers x_r for every $\epsilon>0$. The following weaker form of this statement is proved: If the indefinite form f in n ($\geqq 5$) variables and non-zero determinant represents zero properly and is not a multiple of a rational function, then the inequalities $0<|f|<\epsilon$ are solvable in integers x_r for every $\epsilon>0$. Let $M(f)$ be the lower bound of the non-zero values of $|f|$. Then the preceding statement follows immediately from the following two theorems. (a) If f is a zero form and $M(f)>0$, then there exists a form

$$F=h(X_1X_2+\theta X_3^2+c_3X_3^2+c_4X_4^2+\cdots+c_nX_n^2)$$

such that $M(F)>0$, $c_3, \cdots, c_n$ are integers different from zero, θ is rational or irrational, and F is derived from f by rational, non-singular transformations. (b) If θ is irrational, $c_3, \cdots, c_n$ are integers, and $n\geqq 5$, then the inequalities $0<|X_1X_2+\theta X_3^2+c_3X_3^2+\cdots+c_nX_n^2|<\epsilon$ are solvable in integers X_r for every $\epsilon>0$. The proof makes use of several lemmas, some of the author and some due to Jones [Trans. Amer. Math. Soc. **33**, 92–110, 111–124 (1931)], Ross [Proc. Nat. Acad. Sci. U. S. A. **18**, 600–608 (1932)], Smith [Collected Math. Papers, v. 1, Oxford, 1894, pp. 455–506] and Weyl [Math. Ann. **77**, 313–352 (1916)]. It is conjectured that the main statement remains true even for $n=3$.

E. Grosswald (Philadelphia, Pa.).

Watson, G. L. On indefinite quadratic forms in three and four variables. J. London Math. Soc. **28**, 239–242 (1953).

If $\theta_1, \cdots, \theta_m$ are $m\geqq 1$ nonzero real numbers not of the same sign, whose ratios are not all rational, then one may ask whether for every $\epsilon>0$ the inequality $|\theta_1x_1^2+\cdots+\theta_mx_m^2|<\epsilon$ is soluble in integers $x_1, \cdots, x_m$ not all zero. Davenport and Heilbronn [same J. **21**, 185–193 (1946); these Rev. **8**, 565] proved that the answer is affirmative for $m=5$ (and, therefore, for $m\geqq 5$). Now the author establishes the result for $m=3$ and $m=4$, considering, however, special types of forms only. For $m=3$ he considers forms

$$f(x, y, z)=x^2-a\theta y^2-(a\theta+1)z^2,$$

where a is an arbitrary positive integer and θ denotes the positive root of the equation $\theta^2=a\theta+1$. He then even proves that for all integers $X>0$ the simultaneous inequalities $0<x\leqq X$, $0<y\leqq X$, $0<z\leqq X$, $|f|<CX^{-2}$, where $C=C(a)$ denotes a conveniently chosen constant, have an integer solution x, y, z. For $m=4$ he considers forms of the special kind $f(x, y, z, w)=x^2+dy^2-\theta^2(z^2+dw^2)$, where d is an arbitrary positive integer and θ is any number of some quadratic fields (which depend on d).

J. F. Koksma.

Barnes, E. S. The minimum of a bilinear form. Acta Math. **88**, 253–277 (1952).

Let $B(x, y, z, t)=\alpha xy+\beta xt+\gamma yz+\delta yt$ be a real bilinear form. A form is said to be equivalent to B if it may be obtained from B by a substitution

$$\begin{pmatrix} x & z \\ y & t \end{pmatrix}=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} x' & z' \\ y' & t' \end{pmatrix} \quad\text{or}\quad \begin{pmatrix} x & z \\ y & t \end{pmatrix}=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} z' & x' \\ t' & y' \end{pmatrix},$$

where p, q, r, s are integers such that $ps-qr=\pm 1$. The quantities $\Delta(B)=\alpha\delta-\beta\gamma$ and $\theta(B)=|\beta-\gamma|$ are then equivalent invariants. With the bilinear form $B(x, y, z, t)$ is associated the quadratic form $B(x, y, x, y)$ of discriminant $D=\theta^2-4\Delta$, equivalent bilinear forms yielding equivalent quadratic forms. Define $\omega=\theta/\sqrt{|D|}$.

Let $M(B)$ be the greatest lower bound of $|B(x, y, z, t)|$ over integers x, y, z, t such that $xt-yz=\pm 1$. For symmetric forms, characterized by $\omega=0$, Schur [S.-B. Preuss. Akad. Wiss. **1913**, 212–231] has proved when $D>0$ that $M(B)\leqq D^{1/2}/2\sqrt{5}$, with equality for multiples of exactly one class of equivalent bilinear forms. An analogous result was obtained when $D>0$ for factorizable forms, characterized by $\omega=1$, by Davenport and Heilbronn [Quart. J. Math., Oxford Ser. **18**, 107–121 (1947); these Rev. **9**, 79; see also Barnes, Acta Math. **86**, 323–336 (1951); these Rev. **14**, 142]. In the present paper the author considers bilinear forms with arbitrary ω, and proves that $M(B)\leqq D^{1/2}/2\sqrt{5}$ for all ω when $D>0$, with equality if and only if B is a multiple of any form in a specific countable set of equivalence classes. When $D<0$, he shows that $M(B)\leqq(-D)^{1/2}/2\sqrt{3}$ for all ω, with equality under similar circumstances. He also solves the more difficult question of finding the best possible estimate for $M(B)$ as a function of ω and D for the case $D>0$, $0\leqq\omega\leqq\omega_0=1.24$ (approximately). A continuous function $\chi(\omega)$, whose graph is a zig-zag line, is defined for the indicated range, and a specific set of nine quadratic forms Q_i ($i=0, 1, \cdots, 8$) is given, and the author then proves that $M(B)\leqq D^{1/2}\chi(\omega)/2$ when $D>0$. He shows further that for each ω ($0\leqq\omega\leqq\omega_0$) there exists a form B for which equality holds and for which the quadratic form associated with B is equivalent to a multiple of some Q_i. This theorem includes as special cases the above-mentioned results of Schur, Davenport and Heilbronn.

In proving his results, the author first establishes a connection between $M(B)$ and the quadratic form Q associated with B, and shows easily that

$$M(B)=\tfrac{1}{2}|D|^{1/2}\inf|b\cdot|D|^{-1/2}-\omega|,$$

where b runs over all middle coefficients of forms equivalent to Q. The estimates for $M(B)$ independent of ω are simple consequences of the above formula, but in order to obtain the best possible results for given D and ω, some rather complicated reasoning is used which involves the chain of reduced forms equivalent to Q. The author remarks that his methods do not enable him to evaluate $\chi(\omega)$ for $\omega>\omega_0$.
I. Reiner (Urbana, Ill.).

Barnes, E. S., and Swinnerton-Dyer, H. P. F. The inhomogeneous minima of binary quadratic forms. II. Acta Math. **88**, 279–316 (1952).

Using the methods and notation of Part I [Acta Math. **87**, 259–323 (1952); these Rev. **14**, 730], the authors consider the problem of finding an enumerably infinite number of inhomogeneous minima for the norm-forms x^2-11y^2 and $x^2+xy-3y^2$. Results of this kind have been obtained previously by Davenport [Nederl. Akad. Wetensch., Proc. **49**, 815–821 (1946); **50**, 378–389, 484–491, 741–749, 909–917 (1947); these Rev. **8**, 444, 565; **9**, 79, 412] for the form x^2+xy-y^2, and by Varnavides [ibid. **51**, 396–404, 470–481 (1948); these Rev. **10**, 19] for x^2-2y^2. The authors' results for x^2-11y^2 are analogous to those of Davenport and Varnavides, and include furthermore a proof of the existence for each $\epsilon>0$ of an uncountably infinite set of points P for which $M'>M(f,P)>M'-\epsilon$, where $M'=\lim M_k(f)$. The behavior of the minima of $x^2+xy-3y^2$ is surprisingly different, and the authors show the existence of countably many isolated minima extending below M'. The basic concept in both proofs is that of the set of transforms of a given point by all powers of the fundamental automorph of the given form. The authors conjecture that for any form f, $M_1(f)$ is rational, isolated, and taken at rational points (and possibly also at irrational points in $k(\sqrt{d})$, that $M_2(f)$ exists and is taken at points in $k(\sqrt{d})$, and further that these results are the strongest possible ones true for all forms. The paper concludes with a brief indication of the extension of its methods to norm-forms in n variables for real and complex fields.
I. Reiner (Urbana, Ill.).

Richert, Hans-Egon. Ein Gitterpunktproblem. Math. Ann. **125**, 467–471 (1953).

Let $D(x;q_1,h_1,q_2,h_2)$ denote the number of lattice points u, v in the region $u>0$, $v>0$, $uv\leq x$, which satisfy $u\equiv h_1 \pmod{q_1}$, $u\equiv h_2 \pmod{q_2}$. Further put $x-[x]-\frac{1}{2}=(x)$. From a previous paper [Math. Z. **58**, 71–84 (1953); these Rev. **14**, 845] the author borrows the formula

$$\sum_{0<n\leq y}\psi(n^{-1}x+\omega)=O(1)+O(x^{-1/2}y^{3/2})+O(x^{11/41}y^{8/41}),$$

uniformly in x, y, ω, θ, where $x>0$, $y>0$, ω real, and θ^{-1} is a positive integer. By a well-known classical argument the author derives that

$$D(x;q_1,h_1,q_2,h_2)=X\log X$$
$$-\left(\frac{\Gamma'}{\Gamma}\left(\frac{h_1}{q_1}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{h_2}{q_2}\right)+1\right)X+O(X^{27/82}),$$

where $X=x/(q_1q_2)$, uniformly in x, q_1, q_2, h_1, h_2 $(q_1q_2\leq x)$. The result can be applied to the circle problem, and more generally to the problem of the number of ideals with norm $\leq x$ in a quadratic field. The special case $\omega=0$, $\theta=1$ gives Nieland's result for Dirichlet's divisor problem [Thesis, Groningen, 1933]. The author announces that he will show in a forthcoming paper that the error term in the divisor problem can be reduced to $O(x^{15/46}\log^{12/23}x)$.
N. G. de Bruijn (Amsterdam).

Cole, A. J. A problem of Diophantine approximation. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 144–157 (1953).

It is shown that

$$0.309\cdots\leq \overline{\mathrm{bd}}\ \liminf x|\theta x+\alpha-y|\leq 0.409\cdots,$$

where x, y, are rational integers, θ is irrational and α not of the form $\theta m+n$ (m, n integers). That the lim inf$\leq 5^{-\frac{1}{2}}$ is an old theorem of Khintchine [Math. Ann. **111**, 631–637 (1935)]. In a footnote it is shown that an example of Poitou and Descombes [C. R. Acad. Sci. Paris **234**, 581–583, 1522–1524 (1952); these Rev. **13**, 825, 921] allows $0.309\cdots$ to be replaced by $0.35\cdots$.
J. W. S. Cassels.

Davenport, H. Simultaneous Diophantine approximation. Proc. London Math. Soc. (3) **2**, 406–416 (1952).

Let α, β be given real numbers. What is the greatest lower bound c_0 of all constants $c>0$ for which the simultaneous inequalities $r(p-\alpha r)^2<c$, $r(q-\beta r)^2<c$ have infinitely many integral solutions p, q, r $(r>0)$? It was shown by Furtwängler that $c_0\geq 1/23^{1/2}$ and by Davenport and Mahler that $c_0\leq 2/23^{1/2}$ [Duke Math. J. **13**, 105–111 (1946); these Rev. **7**, 506], whereas Mullender proved

$$c_0\leq 2^7/3^{3/2}23^{1/2}=1/2.4003\cdots$$

by means of the geometry of numbers. Now using in principle Mullender's method the author proves

$$c_0\leq 1/46^{1/4}=1/2.6043\cdots.$$

At the end of the paper the author makes some remarks concerning the possibilities of improving his estimate.
J. F. Koksma (Amsterdam).

LeVeque, W. J. On Mahler's U-numbers. J. London Math. Soc. **28**, 220–229 (1953).

The complex number ξ is called a U-number if for some fixed integer $n\geq 1$ and for all $\omega>0$ there exist infinitely many sets of $n+1$ integers $a_0, a_1, \cdots, a_n$ with $A=\max(|a_0|,\cdots,|a_n|)$ such that

$$0<|a_0+a_1\xi+\cdots a_n\xi^n|<A^{-\omega}.$$

The author calls ξ a U_m-number if this condition is fulfilled for $n=m$ but for no $n\leq m-1$. He proves several theorems on U_m-numbers (U_1-numbers are identical with Liouville numbers), e.g., there exist U_m-numbers for every $m\geq 1$. Also he proves that numbers satisfying a certain transcendency condition due to Th. Schneider are not U_m-numbers for too small values of m. [Cf. Mahler, Nederl. Akad. Wetensch., Proc. **39**, 633–640, 729–737 (1936).]
J. F. Koksma (Amsterdam).

LeVeque, W. J. Note on S-numbers. Proc. Amer. Math. Soc. **4**, 189–190 (1953).

The transcendental number ξ is said to be an S-number if there is a constant $\gamma=\gamma(\xi)>0$ and a sequence of positive constants $\Gamma_m=\Gamma_m(\xi)$ $(m=1,2,\cdots)$ such that for each polynomial $f(x)\equiv a_0+a_1x+\cdots+a_mx^m$ of arbitrary degree m and integer coefficients $a_0,\cdots,a_m$ such that

$$a=\max(|a_0|,\cdots,|a_m|)\geq 1$$

the inequality $|f(\xi)| \geqq \Gamma_m a^{-\gamma m}$ holds. Now let γ_r denote the infimum of all numbers γ' such that almost all real ξ are S-numbers with $\gamma \leqq \gamma'$ and let γ_c denote the infimum of all γ' such that almost all complex numbers ξ are S-numbers with $\gamma \leqq \gamma'$. Mahler conjectured that $\gamma_r = 1$, $\gamma_c = 1/2$ and he proved $\gamma_r \leqq 4$, $\gamma_c \leqq 4$ [Math. Ann. **106**, 131–139 (1932)]; the reviewer proved $\gamma_r \leqq 3$, $\gamma_c \leqq 5/2$ [Monatsh. Math. Phys. **48**, 176–189 (1939); these Rev. **1**, 137]; Kubilyus proved the conjecture in the special case $m=2$ [Doklady Akad. Nauk SSSR (N.S.) **67**, 783–786 (1949); these Rev. **11**, 82]. In this paper the author, combining Mahler's original argument with a theorem of Fel'dman [Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 53–74 (1951); these Rev. **12**, 595], proves $\gamma_r \leqq 2$, $\gamma_c \leqq 3/2$.
J. F. Koksma (Amsterdam).

Mahler, K. On the approximation of π. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 30–42 (1953).

The author derives the remarkable inequality

$$|\pi - p/q| > q^{-42}$$

for all positive integers $p, q \geqq 2$. This measure of irrationality for π is stronger than any previous result in this direction. Furthermore, he gives the following measure of transcendency for π. Let ω be a real or complex algebraic number. Denote by R the rational field K if ω is real, and the Gaussian imaginary field $K(i)$ if ω is non-real. Further denote by ν the degree of ω over R, by

$$a_0 z^\nu + a_1 z^{\nu-1} + \cdots + a_\nu = 0 \quad (a_0 \neq 0),$$

an equation for ω with integral coefficients in R which is irreducible over this field. Put

$$m = [20 \cdot 2^{3(\nu-1)/2}],$$
$$a = \max(|a_0|, |a_1|, \cdots, |a_\nu|, (m+1)^{(m+1)/\nu}).$$

Then

$$|\pi - \omega| > e^{m+1}(m+1)^{-(m+1)} a^{-(m+1)\nu \log(m+1)}.$$

This measure of transcendency is compared with a similar result of Fel'dman [Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 53–74 (1951); these Rev. **12**, 595]. The paper closes with some applications. The proofs depend on a basic result on a system of approximation forms of the type

$$\sum_{k=0}^{m} A_{hk}(x)(\log x)^k \quad (h = 0, 1, \cdots, m+1)$$

where the A's are polynomials with integral coefficients, obtained in an earlier paper [Philos. Trans. Roy. Soc. London. Ser. A. **245**, 371–398 (1953); these Rev. **14**, 624].
J. Popken (Utrecht).

Čudakov, N. G. On algebraic independence of values of the exponential function. Ukrain. Mat. Žurnal **3**, 211–217 (1951). (Russian)

A new variant of the Hilbert-Hurwitz proof of Lindemann's theorem, using the language of modern algebra.
K. Mahler (Manchester).

Churchhouse, R. F. A criterion for irrationality. Canadian J. Math. **5**, 253–260 (1953).

Using a criterion of Legendre for the irrationality of numbers represented by continued fractions the author proves that the continued fraction $K(x^{\psi(n)}/1)$ assumes irrational values for $x = r/s$, where r and s are positive integers with $(r, s) = 1$ and such that $r < s^\gamma$. Here $\psi(n)$ is a strictly increasing positive integral-valued function of n and

$$\gamma = \liminf \frac{1}{\psi(n)} \sum_{r=0}^{n-1} (-1)^r \psi(n-r).$$

The result is applied to a number of special cases.
W. J. Thron (St. Louis, Mo.).

ANALYSIS

Azpeitia, A. G. Note on the mean value theorem. Gaceta Mat. (1) **4**, 9–10 (1952). (Spanish)

For a function $f(x)$ continuous in a closed and bounded interval (a, b), and having a derivative at all except at most a finite number of these points, by application of the law of the mean to each of the resulting subintervals the author shows that

$$\inf f'(x) \leqq \frac{f(b) - f(a)}{b - a} \leqq \sup f'(x).$$

It is pointed out that the result holds more generally if the set of points where $f'(x)$ fails to exist has a finite or null set of limit points. *E. F. Beckenbach* (Los Angeles, Calif.).

Thompson, Lee Detmer. Converse of a well known theorem on integral means. Proc. Amer. Math. Soc. **4**, 402–407 (1953).

If a function $f(x)$ is continuous, its mean $M_h(x)$ over the interval $(x, x+h)$ has a continuous first derivative. If f belongs to class $C^{(n)}$, then M_h belongs to class $C^{(n+1)}$. The present paper is devoted to studying the converses to these results. A direct converse is not possible as is shown by the example of a function f equal almost everywhere to a continuous function. Under the additional hypothesis that f is mean continuous, that is, satisfies the condition $\lim_{h \to 0} M_h(x) = f(x)$, the converse is proved: if $M_h^{(n+1)}$ is continuous, then $f^{(n)}$ is continuous. *J. W. Green.*

Bellman, Richard. On an inequality due to Weinberger. Amer. Math. Monthly **60**, 402 (1953).

See the review of Weinberger's paper [Proc. Nat. Acad. Sci. U. S. A. **38**, 611–613 (1952); these Rev. **14**, 24].

Gonçalves, J. Vicente. Sur le reste de la formule de Taylor. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **2**, 89–90 (1952).

Gonçalves, J. Vicente. Sur un développement de $f(x, y)$. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **2**, 91–92 (1952).

Moppert, K.-F. Über einen verallgemeinerten Ableitungsoperator. Comment. Math. Helv. **27**, 140–150 (1953).

The author discusses a definition of differentiation of arbitrary real order α which he attributes to A. K. Grünwald [Z. Math. Phys. **12**, 441–480 (1867)] and which somewhat resembles that given by N. Stuloff [Math. Ann. **122**, 400–410 (1951); these Rev. **12**, 680]. Set

$$D^\alpha f|_a^b = \lim_{n \to \infty} \left(\frac{n}{b-a}\right)^\alpha \sum_{\nu=0}^{n-1} (-1)^\nu \binom{\alpha}{\nu} f[b - \nu(b-a)n].$$

The principal theorem is that (i) if $\alpha<0$, if $f(x)$ is differentiable in $a<x\leqq b$, and if $f(x)=O((x-a)^{-\beta})$, $x\to a$, $\beta<1$, then $D^\alpha f|_a^b$ exists and has the value

$$\frac{1}{\Gamma(-\alpha)}\int_a^b (b-x)^{-\alpha-1}f(x)dx;$$

(ii) if $\alpha>0$ and α is not an integer, if $f(x)$ has a continuous $([\alpha]+1)$th derivative in $a<x\leqq b$, and

$$f^{([\alpha]+1)}(x)=O((x-a)^{-\beta-1}),\quad \beta<1,$$

then $D^\alpha f|_a^b$ exists and is equal to the $([\alpha]+1)$th derivative at b of

$$\frac{1}{\Gamma(1+[\alpha]-\alpha)}\int_a^b (b-x)^{[\alpha]-\alpha}f(x)dx.$$

It is to be noted that when $\alpha<1$, the definition applies to functions which may become infinite at the point a; this is essential if the operator is to have the expected formal properties, since (for example) the αth derivative of a constant does become infinite at $x=a$. The author gives several additional properties of the operator. *R. P. Boas, Jr.*

Freud, Géza. Restglied eines Tauberschen Satzes. II. Acta Math. Acad. Sci. Hungar. **3** (1952), 299–307 (1953). (Russian summary)

In paper I, the author [same Acta **2**, 299–308 (1951); these Rev. **14**, 361] made appropriate hypotheses on the Laplace transform $\int_0^\infty f(t)e^{-st}d\tau(t)$ and obtained an estimate of the function $\int_0^x t^\beta f(t)d\tau(t)$. Under the same hypotheses, the present paper gives an estimate of the m-fold integral of the latter function, m being a positive integer.

R. P. Agnew (Ithaca, N. Y.).

Rajagopal, C. T. On a one-sided Tauberian theorem; a further note. J. Indian Math. Soc. (N.S.) **17**, 33–42 (1953).

The author [same J. (N.S.) **16**, 47–54 (1952); these Rev. **14**, 160] obtained a bound for $\int_0^h a(u)du$ from a bound for $\int_0^\infty \varphi(ut)a(u)du$ and the hypothesis $ua(u)\geqq -W$. This further note replaces the last hypothesis by

$$u(\log\log u)a(u)\geqq -W$$

and obtains analogous results. The case in which $\varphi(t)=e^{-t}$ yields results of Pennington [Proc. Amer. Math. Soc. **3**, 557–565 (1952); these Rev. **14**, 159]. *R. P. Agnew.*

Rajagopal, C. T. A generalization of Tauber's theorem and some Tauberian constants. Math. Z. **57**, 405–414 (1953).

The main theorem involves lengthy hypotheses and notations but is proved very simply. It involves transformations of the form

$$\Psi(x)=\int_0^\infty \psi(x,u)A(u)du$$

applied to functions $A(u)$ satisfying a Tauberian condition of Schmidt type and gives an estimate of $|\Psi(x)-A(x')|$ when x and x' are related in a specified way. One application involves transformations of the form

$$\Psi^*(x)=x^{-1}\int_0^\infty \psi^*(u/x)A(u)du$$

where $\psi^*(u)\geqq 0$, $\int_0^\infty\psi(u)du=1$, and $\int_0^\infty\psi^*(u)|\log u|du<\infty$. If $A(u)$ has bounded variations over each finite interval $0\leqq u\leqq u_0$ and satisfies the Schmidt condition,

$$\limsup_{u\to\infty}\ \sup_{u\leqq u'\leqq\lambda u}|A(u')-A(u)|\leqq K\log\lambda,$$

when $\lambda>1$, then

$$\limsup_{x\to\infty}|\Psi^*(x)-A(\alpha x)|\leqq K\int_0^\infty \alpha\psi^*(\alpha u)|\log u|du$$

when $\alpha>0$. The latter result is reformulated in terms of the function $\varphi^*(x)=\int_u^\infty \psi^*(x)dx$. Special applications to the transformations of Riesz, Laplace-Abel, Stieltjes, and Lambert are noted. The main theorem and further arguments are used to prove the following theorem on Borel transforms. Let $\sum a_n$ be a series for which $\limsup|n^{1/2}a_n|=L<\infty$ and let $A(u)=\sum_{k\leqq u}a_k$. Let $B(x)=e^{-x}\sum_{k=0}^\infty(x^k/k!)u_k$. Then

$$\limsup_{x\to\infty}|B(x)-A(x)|\leqq(2/\pi)^{1/2}L$$

and, moreover, the constant $(2/\pi)^{1/2}$ is the least constant for which the assertion is valid. *R. P. Agnew.*

Obrechkoff, Nikola. Sur quelques classes de fonctions et de suites. C. R. Acad. Bulgare Sci. **4** (1951), no. 2–3, 1–4 (1953). (Russian. French summary)

The author gives simple proofs of Bernstein's representation for a completely monotonic function and Hausdorff's representation for a completely monotonic sequence. Essentially the same proof of Bernstein's theorem was, as the author observes, given independently by B. I. Korenblyum [Uspehi Matem. Nauk (N.S.) **6**, no. 4(44), 172–175 (1951); these Rev. **13**, 329]. *R. P. Boas, Jr.* (Evanston, Ill.).

Obrechkoff, Nikola. Sur quelques classes de fonctions réelles. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie I. **47**, 237–258 (1951). (Bulgarian. French summary)

This paper is summarized in the paper reviewed above.

R. P. Boas, Jr. (Evanston, Ill.).

Bilharz, Herbert. Bemerkung zur genäherten Quadratur. Arch. Math. **3**, 251–256 (1952).

The author estimates asymptotically the coefficients of the remainder terms in the approximate integration formulae obtained by integrating Newton's interpolation formulae for equidistant points, for ordinary as well as central interpolation. Thus by integration of the forward difference formula he obtains

$$\int_0^1 f(x)dx=f(0)+\sum_{\nu=1}^{n-1}A_\nu f(0,1,\cdots,\nu)+\frac{A_n}{n!}f^{(n)}(\eta)\qquad(0<\eta<n-1),$$

where

$$A_\nu=\int_0^1 x(x-1)\cdots(x-\nu+1)dx.$$

By expressing here the integrand as a quotient of values of the gamma function and by applying an Abelian theorem for Laplace integrals the author obtains the asymptotic estimate

$$(1)\qquad (-1)^{n-1}A_n/n!\sim 1/(n\log^2 n)\quad\text{as}\quad n\to\infty.$$

Similar results are obtained for the remainder of Gauss' central difference formula. In expressing A_n asymptotically

as a Laplace integral this reviewer obtains

$$(-1)^{n-1}\frac{A_n}{n!}=\int_0^1 \frac{x\Gamma(n-x)\Gamma(x)}{\Gamma(n+1)}\frac{\sin \pi x}{\pi}dx$$

$$\sim\frac{1}{n}\int_0^1 e^{-x\log n}x\Gamma(x)\frac{\sin \pi x}{\pi}dx$$

instead of the author's

$$(-1)^{n-1}\frac{A_n}{n!}=\int_0^1 \frac{\Gamma(n-x)}{\Gamma(n+1)\Gamma(x)}dx\sim\frac{1}{n}\int_0^1 e^{-x\log n}\frac{dx}{\Gamma(x)},$$

both expressions, however, leading to the same result (1).

I. J. Schoenberg (Philadelphia, Pa.).

Calculus

★Abdelhay, J. Curso de análise matemática. Vol. I. [Course of mathematical analysis. Vol. I.] 2d ed. Universidade do Brasil, Rio de Janeiro, 1953. xvi+232 pp.

This is a textbook covering what in the U. S. A. would be part of a course in advanced calculus and part of an introductory course in real variables. The topics covered are the topology of the real line; limits, continuity and differentiation for functions of one real variable; Riemann integrals (in considerable detail); calculation of indefinite integrals; improper integrals. *R. P. Boas, Jr.* (Evanston, Ill.).

★Kaplan, Wilfred. Advanced calculus. Addison-Wesley Press, Inc., Cambridge, Mass., 1952. xiii+679 pp. \$8.50.

Introduction. Review of algebra, analytic geometry and calculus. Chap. 1. Vectors. Chap. 2. Differential calculus of functions of several variables. Chap. 3. Vector differential calculus. Chap. 4. Integral calculus of functions of several variables. Chap. 5. Vector integral calculus. Chap. 6. Infinite series. Chap. 7. Fourier series and orthogonal functions. Chap. 8. Ordinary differential equations. Chap. 9. Functions of a complex variable. Chap. 10. Partial differential equations. *Table of contents.*

★Kaplan, Wilfred. A first course in functions of a complex variable. Addison-Wesley Publishing Co., Inc., Cambridge, Mass., 1953. vii+485–619 pp. \$3.50.

Reprint of chap. 9 of the book listed above.

Andress, W. R. The expansion of a function in terms of its values and derivatives at several points. Amer. Math. Monthly 60, 394–396 (1953).

The author uses contour integration to give a complex-variable treatment of the formula discussed in the real domain by Hummel and Seebeck [same Monthly 56, 243–247 (1949); these Rev. 10, 516]. *R. P. Boas, Jr.*

Good, I. J. A generalisation of Dirichlet's multiple integral. Edinburgh Math. Notes 38, 7–8 (1952).

This is a generalization of Dirichlet's formula

$$\int\cdots\int f(t_1+\cdots+t_n)t_1^{\alpha_1-1}\cdots t_n^{\alpha_n-1}dt_1\cdots dt_n$$

$$=\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\cdots+\alpha_n)}\int_0^1 f(\tau)\tau^{\alpha_1+\cdots+\alpha_n-1}d\tau,$$

$t_1+\cdots+t_n\leqq 1$ [Whittaker and Watson, Modern analysis, 4th ed., Cambridge, 1915, section 12.5]. $f_1(t),\cdots,f_n(t)$ is a sequence of Lebesgue measurable functions on $0\leqq t\leqq 1$; $M_r=m_1+\cdots+m_r$, $X_r=x_1+\cdots+x_r$, $x_r\geqq 0$, m_r real, $m_{n+1}=0$, $r=1, 2, \cdots, n$. Then if $X_n\leqq 1$,

$$\int\cdots\int\prod_{r=1}^{n-1}\left\{x_r^{m_r}f_r\left(\frac{X_r}{X_{r+1}}\right)\right\}x_n^{m_r}f_n(X_n)dx_1\cdots dx_n$$

$$=\prod_{r=1}^{n}\int_0^1 f_r(x)(1-x)^{m_{r+1}}x^{M_r+r-1}dx$$

provided the integrals on the right all exist. The proof is by induction after changing the variables from $x_1, \cdots, x_n$ to $x_1, \cdots, x_{n-1}, X_n$ and $x_r=X_ny_r$ $(r=1, 2, \cdots, n-1)$. A special case is $f_r(t)=t\Lambda_r$, $\Lambda_r=\lambda_1+\cdots+\lambda_r$, $m_r>-1$, $\lambda_r+m_r>-1$.

Then if $X_n<1$,

$$\int\cdots\int\prod_{r=1}^{n}\{x_r^{m_r}X_r^{\lambda_r}dx_r\}=\prod_{r=1}^{n}\frac{\Gamma(\Lambda_r+M_r+r)\Gamma(m_{r+1}+1)}{\Gamma(\Lambda_r+M_{r+1}+r+1)}.$$

R. L. Jeffery (Kingston, Ont.).

Peyovitch, T. Contribution à l'étude de la formule

$$\underbrace{\int_a^\infty dx\int_a^\infty dx\cdots\int_a^\infty}_{n} f(x)dx=\frac{1}{(n-1)!}\int_a^\infty (t-x)^{n-1}f(x)dx.$$

Bull. Soc. Math. Phys. Serbie 4, no. 3–4, 7–10 (1952). (Serbo-Croatian summary)

Natucci, A. L'uso del principio d'induzione nel calcolo di certi integrali. Giorn. Mat. Battaglini (5) 1 (81), 85–87 (1952).

Roy, S. N. Some useful results in Jacobians. Calcutta Statist. Assoc. Bull. 4, 117–122 (1952).

Let $F_i(y_1, \cdots, y_m, x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n})=0$, $i=1, \cdots, m+n$, be a system of $m+n$ real equations possessing real solutions

$$y_j=f_j(x_1, \cdots, x_m), \quad j=1, \cdots, m;$$
$$x_k=g_k(x_1, \cdots, x_m), \quad k=m+1, \cdots, m+n.$$

Assuming the usual conditions, including the non-vanishing of the Jacobians, the writer shows that

$$\left|\frac{\partial(f_1, \cdots, f_m)}{\partial(x_1, \cdots, x_m)}\right|=\left|\frac{\partial(F_1, \cdots, F_{m+n})}{\partial(x_1, \cdots x_m, x_{m+1}, \cdots x_{m+n})}\right|$$
$$\div\left|\frac{\partial(F_1, \cdots, F_{m+n})}{\partial(y_1, \cdots, y_m, x_{m+1}, \cdots, x_{m+n})}\right|.$$

P. V. Reichelderfer (Columbus, Ohio).

Balasubramanian, N. Some identities of operators and their applications. Math. Student 20, 74–76 (1952).

Let $D=d/dx$ and $\theta=xD$. The iterated operator θ^r is expressed as a linear combination of the operators x^sD^s, where r and s are positive integers. Applications include the representation of n^r, where n is any real number, by a linear combination of the functions $n(n-1)\cdots(n-i)$ where $i=0, 1, 2, \cdots, (r-1)$. From the latter representation a formula for $\sum_{i=1}^n i^r$ is written. Some generalizations are considered. *R. V. Churchill* (Ann Arbor, Mich.).

Theory of Sets, Theory of Functions of Real Variables

***Kuratowski, Kazimierz, i Mostowski, Andrzej. Teoria mnogosci. (Theory of sets.)** Monografie Matematyczne, Tom XXVII. Polskie Towarzystwo Matematyczne, Warszawa-Wrocław, 1952. ix+311 pp.

This excellent monograph presents a modern course on general set-theory. The presentation follows a mimeographed course of lectures by Kuratowski on set-theory, issued in 1924, but is considerably enlarged and completed. Thus the systematic use of logical operators in set-theory, the study of the operation of the direct product of sets, and the theory of partial order form new features of this volume. Chapter VI, dealing with the study of independence and non-contradiction of the axioms of set-theory, is new and contains a survey of Gödel's work and recent results connected with it.

The book is extremely readable, due to a system of development of set-theory which combines the two approaches: the "naive" method followed by Cantor himself, and the formalistic treatment developed on the axiomatic method. The symbolism of mathematical logic is used throughout, but with moderation, and ample motivation is given in the text appealing to the intuition on the infinite sets. This intuition will continue to be necessary as no one system of axioms now known can claim to express the full intuitive freedom of construction available to "naive" set-theory. It is this combination of methodologies that makes the book unique as a monograph on set-theory.

The division of the book is as follows: Chap. I. Algebra of sets; Chap. II. Relations, functions, infinite operations; Chap. III. Theory of power of sets (i. e., cardinal numbers, their arithmetic, etc.); Chap. IV. Ordered sets; Chap. V. Well-ordered sets; Chap. VI. Non-contradictory character and independence of axioms. There is a supplement on the "paradoxical" decompositions of the sphere (results of Hausdorff-Banach-Tarski, including the recent contribution to the problem by Robinson and Sierpinski). Some illustrations of the fundamental role of set-theory in topology and in algebra as well as in measure theory are given in the text.

The book is written with great clarity and—apart from the interest which it has as a "Monograph"—its didactic merits are such as to make it very usable as an undergraduate text. The more advanced sections are especially marked and can be omitted without destroying the continuity of the presentation. *S. Ulam.*

Watanabe, Hideaki. Sur une séparation des ensembles analytiques plans par une courbe mesurable (*B*). Proc. Japan Acad. 26, no. 7, 17–20 (1950).

A theorem concerning the separation of two analytical sets in a plane of which one is located below the other with respect to a curve measurable (*B*) is established. This result leads in particular to an alternate proof of the theorem of Lusin [Leçons sur les ensembles analytiques, Gauthier-Villars, Paris, 1930, p. 234]. *P. Nesbeda.*

Sierpiński, Wacław. Sur quelques conséquences du théorème de M. Kondô concernant l'uniformisation des complémentaires analytiques. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **44** (1951), 56–62 (1952).

Using a theorem of Kondô the author proves various theorems on the projective classes of sets. Among others he proves that if $U_1, U_2, \cdots$ is an infinite sequence of sets all belonging to the class $C(A)$ or $PC(A)$, there exists an infinite sequence of disjoint sets $V_n \subset U_n$ belonging to the same class so that $\sum_{n=1}^{\infty} V_n = \sum_{n=1} U_n$. *P. Erdös.*

Mrówka, Stanisław. Sur une propriété des ensembles fermés et bornés. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **44** (1951), 76–77 (1952).

Sierpiński proved that if E is a closed and bounded set in the plane, there exist $2^{\aleph_0}$ straight lines which intersect E in precisely one point. The author proves the following conjecture of Zarankiewicz: Let E be a closed and bounded set in n-dimensional space; then there exist $2^{\aleph_0}$ hyperplanes of dimension $n-1$ which intersect E in precisely one point.
P. Erdös (South Bend, Ind.).

Neves Real, Luís. From the rational numbers to the real numbers. Bol. Soc. Portuguesa Mat. Sér. A. **1**, 59–135 (1951). (Portuguese)

This is an expository paper. The algebraic and topological properties of the sets of rational and real numbers are well presented. The real numbers are introduced according to the Cantor model. *P. Nesbeda* (Camden, N. J.).

Kurepa, Đuro. The problem of measure and monotonic mappings of partially ordered sets. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **277**, 229–237 (1950). (Serbo-Croatian)

Besides some expository remarks, this paper contains a generalization of a theorem of Ulam on the non-existence of certain measures [Fund. Math. 16, 140–150 (1930)] to the case of measures whose values lie in an arbitrary partially ordered set. *E. Hewitt* (Seattle, Wash.).

Hartman, S. Zur Gitterpunktverteilung bei Verschiebungen von Mengen. Studia Math. **13**, 87–93 (1953).

Let E be a given set of finite measure in R^n. For any x in R^n let $g(x)$ be the number of points with integral coordinates contained in $x+E$. Let $\psi_s(x)$ be the characteristic function of the set where $g(x)=s$, and let M_s be the part of this set contained in the unit cube W. Let ξ in R^n be a point whose coordinates are rationally independent mod 1. Write $F(x)=\lim_{N\to\infty} N^{-1}\sum_{k=1}^{N} g(x+k\xi)$, and let $F_s(x)$ denote the corresponding limit with g replaced by ψ_s. As an application of the ergodic theorem it is shown that a. e. (i) $F(x)=m(E)$, (ii) $F_s(x)=m(M_s)$, and (iii) $\sum_1^\infty sF_s(x)=m(E)$. In case E is Jordan measurable, Weyl's theorem shows that the exceptional sets are all empty. It may be noted that (iii) follows directly from (ii) and the fact that $\int_W g=m(E)$.
J. C. Oxtoby (Bryn Mawr, Pa.).

***Bourbaki, N. Eléments de mathématique. XIII. Première partie: Les structures fondamentales de l'analyse. Livre VI: Intégration. Chapitre I: Inégalités de convexité. Chapitre II: Espaces de Riesz. Chapitre III: Mesures sur les espaces localement compacts. Chapitre IV: Prolongement d'une mesure; espaces L^p.** Actualités Sci. Ind., no. 1175. Hermann et Cie, Paris, 1952. ii+237+v pp. 2500 francs.

This volume presents the first chapters of a long awaited Bourbaki book. The author's point of view was in part already known through its concise presentation given by H. Cartan [Bull. Soc. Math. France **69**, 71–96 (1941); these Rev. **7**, 447] and R. Godement [Trans. Amer. Math. Soc. **63**, 1–84 (1948); these Rev. **9**, 327]. As in other Bourbaki books, the topics are well organized and nicely presented, with an aim to great generality whenever it seems justified by applications within mathematics. With this in mind, the

author has included in the text only those notions and results which are fundamental and of interest to a non-specialist. Several useful or important facts, which an expert certainly regrets not to see in the text, are stated as exercises (with hints) together with a large number of exercises in the more common sense of the word. Although only very few results and concepts from the theory of topological vector spaces are used in this volume (the Hahn-Banach theorem and the notion of dual vector spaces), reference is often made to Bourbaki's Espaces vectoriels topologiques, Chap. I and II. [Actualités Sci. Ind., no. 1189, Hermann, Paris, 1953; these Rev. **14**, 880] and to chapters not available as yet. The present volume is elementary and preparatory in the sense that the deep results of the theory will come up in the forthcoming chapters. Bourbaki's attitude towards integration theory seems to be the following one. Firstly, what is essentially of interest is integration on locally compact spaces. In situations encountered in the so-called abstract theory (the most up to date version of which, in book form, is P. R. Halmos' Measure theory [Van Nostrand, New York, 1950; these Rev. **11**, 504]) either there must be a good locally compact topology behind the topology-free considerations, or whenever one has the impression of integrating on a non-locally compact topological space, a suitable device (such as restricting the support of the measure, or the like) reduces the case to the locally compact one. Anyhow, thanks to the known work of Kakutani on abstract L-spaces, integrating in the abstract sense is isomorphic to integrating on a certain immense locally compact space. Secondly, although functions can be built up from sets, as operators in Hilbert space are constructed from projections, the linear functional point of view followed by Bourbaki is considered as technically superior to the set function point of view. It is also essential if one aims at generalizing measure theory and going into the recent theory of distributions [L. Schwartz, Théorie des distributions, tomes I et II, Hermann, Paris, 1950, 1951; these Rev. **12**, 31, 833]. In spite of this, Bourbaki is not completely against the abstract (i.e., topology-free) approach, as one might think from what has just been said. As a matter of fact, Chap. IV of the present volume includes several sections in small print devoted to "mésures abstraits".

The very short Chap. I (pp. 9–16) gives a general version of classical inequalities. Let E be a set, P the vector space of all positive finite real functions on E, M a positive real functional on P such that: $M(0)=0$; $M(\lambda f)=\lambda M(f)$ if $0<\lambda<\infty$; $h\leqq f+g\rightarrow M(h)\leqq M(f)+M(g)$. Let $\varphi(t_1, \cdots, t_n)$ be a finite real function, defined and continuous for $t_i\geqq 0$ such that:

$$t_1>0, \cdots, t_n>0 \text{ imply } \varphi(t_1, \cdots, t_n)>0;$$
$$\varphi(\lambda t_1, \cdots, \lambda t_n)=\lambda\varphi(t_1, \cdots, t_n), \quad 0<\lambda<\infty;$$

the set $\{(t_1, \cdots, t_n);\ t_1\geqq 0, \cdots, t_n\geqq 0,\ \varphi(t_1, \cdots, t_n)\geqq 1\}$ is convex. Then $M(\varphi(f_1, \cdots, f_n))\leqq\varphi(M(f_1), \cdots, M(f_n))$ for $f_1, \cdots, f_n \in P$. In particular, one gets Hölder's inequality $M(f^\alpha g^\beta)\leqq M(f)^\alpha M(g)^\beta$ where $\alpha>0$, $\beta>0$, $\alpha+\beta=1$; and Minkowski's inequality $N_p(f+g)\leqq N_p(f)+N_p(g)$, where $N_p(f)=M(f^p)^{1/p}$ and $1\leqq p<\infty$. A few consequences and a brief historical note complete this chapter.

Chap. II (pp. 17–40) presents the material on ordered vector spaces strictly needed in the sequel. A Riesz space is what in the standard terminology is called a vector lattice. The author's terminology is partly justified as a great deal of what is done in this chapter, or in the book, is due to, or inspired by, F. Riesz. The main theorems, however, refer to complete vector lattices. A band in such a lattice E is a vector subspace $B\subset E$ such that: $x\in B$, $y\in E$, $|y|\leqq|x|$ implies $y\in B$, where $|x|=\sup(x, -x)$; and if $X\subset B$ is bounded from above in E, then $\sup X$ in E lies in B. Riesz's abstract form of Lebesgue's classical decomposition of a function of bounded variation in the line into an absolutely continuous component and a singular one is given as follows. If $A\subset E$, the set A' of all $x'\in E$ such that $\inf(|x|, |x'|)=0$ for all $x\in A$ is a band. $A''=(A')'$ is equal to the least band containing A. The lattice E is the ordered direct sum of its vector subspaces A' and A'', i.e., every $t\in E$ is uniquely $t=t'+t''$ with $t'\in A'$, $t''\in A''$, and $t\geqq 0\rightleftarrows t'\geqq 0$ and $t''\geqq 0$. The least band A'' containing a given set $A\subset E$, or a single $a\in E$, is also described explicitly. The next main result in the chapter is that, in a Riesz space E, a linear functional L can be written as a difference $L=U-V$ of two positive linear functionals U, V if and only if L is bounded, i.e., $\{L(y);\ |y|\leqq x\}$ is a bounded set of real numbers for any positive $x\in E$. The vector space of all bounded linear functionals on E is a complete vector lattice to which the preceding results are applied.

Chap. III (pp. 41–102) begins with the main subject of the book. The starting point of Bourbaki's presentation of integration theory is Riesz's classical theorem [see F. Riesz and B. Sz.-Nagy, Leçons d'analyse fonctionelle, Akadémiai Kiadó, Budapest, 1952; these Rev. **14**, 286], generalized to regular Borel measures on compact or locally compact spaces by Markoff, von Neumann, Kakutani, and others. We mean the theorem that the linear functionals μ on the vector space of all finite numerical continuous functions on the line interval $[a, b]$ given by a Stieltjes integral $\mu(f)=\int_a^b f(x)d\mu(x)$ are those continuous with respect to uniform convergence of functions on $[a, b]$. For didactical reasons, a (real) measure is defined, to start with on a compact space E, as a linear functional $\mu: f\rightarrow\mu(f)=\int f(x)d\mu(x)$ continuous on the Banach space $\mathfrak{C}=\mathfrak{C}(E)$ of all finite real continuous functions on E. The Banach space $\mathfrak{M}=\mathfrak{M}(E)$ dual to $\mathfrak{C}$ of all measures on E is a complete vector lattice. It is noticed that a linear functional on a dense vector subspace $V\subset\mathfrak{C}$ positive on V gives a unique positive measure. A measure on a locally compact space E is next introduced as a linear functional $\mu: f\rightarrow\mu(f)=\int f(x)d\mu(x)$ on the vector space $\mathfrak{K}=\mathfrak{K}(E)$ of all finite real continuous functions on E with compact supports, i.e., vanishing outside compact sets, subject to the following continuity condition: for every compact $K\subset E$, the restriction of μ to the vector subspace $\mathfrak{K}(E, K)\subset\mathfrak{K}(E)$ of those $f\in\mathfrak{K}(E)$ vanishing outside K is continuous with respect to the topology of uniform convergence on E. One can consider $\mathfrak{K}(E)$ as an inductive limit of the Banach spaces $\mathfrak{K}(E, K)$, K variable [J. Dieudonné and L. Schwartz, Ann. Inst. Fourier Grenoble **1**, 61–101 (1950); these Rev. **12**, 417] by putting on $\mathfrak{K}(E)$ the largest topology of locally convex vector space inducing (in this case) on each $\mathfrak{K}(E, K)$ its norm topology. The measures on E are then the linear continuous functionals on $\mathfrak{K}(E)$. The measures on E form a complete vector lattice $\mathfrak{M}=\mathfrak{M}(E)$. Every positive linear functional on a "positively rich" vector subspace of $\mathfrak{K}$ gives a unique positive measure. $\mathfrak{M}$ is also identical to the set of all bounded linear functionals on the Riesz space $\mathfrak{K}$. The expressions $\mu=\mu^+-\mu^-$, $\int fd|\mu|$, $\int fd\mu^+$, $\|\mu\|=\sup\{|\mu(f)|;\ f\in\mathfrak{K},\ \|f\|\leqq 1\}$ are discussed. μ is said to be bounded when $\|\mu\|<\infty$ or, equivalently, when μ is the restriction to $\mathfrak{K}$ of a (unique) linear functional continuous on the Banach space $\overline{\mathfrak{K}(E)}$ of all finite real continuous functions on E vanishing at infinity. The vague topology

is introduced on $\mathfrak{M}$ by taking $\{\mu; |\mu(f)-\mu_0(f)|<\epsilon\}$ as a subbasic neighborhood of $\mu_0 \in \mathfrak{M}$, where $\epsilon>0$ and $f \in \mathcal{K}$. Every vaguely bounded set $H \subset \mathfrak{M}$ (i.e., such that $\sup\{|\mu(f)|; \mu \in H\}<\infty$ for every $f \in \mathcal{K}$) is relatively vaguely compact. Induced measures on open subsets $G \subset E$ are defined. The localization principle motivated by the differentiable manifold situation and supports of measures are presented. Next comes a section on weak vector-valued integrals. E being locally compact, μ a measure on E, F a real locally convex Hausdorff vector space, let $\mathcal{K}_F=\mathcal{K}_F(E)$ be the vector space of all continuous functions on E with values in F vanishing outside compact sets. If F' is the vector space of all linear continuous functionals on F and F'^* the vector space of all linear functionals on F', every $\mathbf{f} \in \mathcal{K}_F$ has a weak vector-integral $\int \mathbf{f} d\mu$, namely, the element of F'^* defined by $\langle \int \mathbf{f}(x)d\mu(x), z' \rangle = \int \langle \mathbf{f}(x), z' \rangle d\mu(x)$, z' being arbitrary in F' and $\langle\ ,\ \rangle$ denoting the bilinear functionals arising from the natural dualities of F'^* with F' and of F with F'. When the closed convex hull of every compact set in F is compact, $\int \mathbf{f} d\mu$ can be interpreted as an element of F under the natural imbedding $F \rightarrow F'^*$. The linear mapping $\mathbf{f} \rightarrow \int \mathbf{f} d\mu$ from $\mathcal{K}_F$ into F is then characterized by: a) if $\mathbf{a} \in F$ and $g \in \mathcal{K}_F$, then $\int \mathbf{a} g d\mu = \mathbf{a} \int g d\mu$; b) for every compact $K \subset E$, the restriction of $\mathbf{f} \rightarrow \int \mathbf{f} d\mu$ to the vector subspace $\mathcal{K}_F(E, K) \subset \mathcal{K}_F(E)$ of all continuous functions $E \rightarrow F$ vanishing outside K is continuous under the topology of uniform convergence for functions on E. If $A \subset F$ is compact, the set of all barycenters $\int \mathbf{x} d\mu(\mathbf{x})$ of A, μ being an arbitrary positive measure on A with $\int d\mu=1$, is shown to coincide with the closed convex hull of A. This chapter ends with product measures. E and F being locally compact spaces, λ and μ measures on E and F, respectively, there is a unique measure $\lambda \otimes \mu$ on $E \times F$ such that $\int ghd(\lambda \otimes \mu) = \int g d\lambda \cdot \int h d\mu$ for $g \in \mathcal{K}(E)$ and $h \in \mathcal{K}(F)$, called the product measure. The bilinear mapping $(\lambda, \mu) \rightarrow \lambda \otimes \mu$ is shown to be vaguely continuous on every vaguely bounded subset of $\mathfrak{M}(E) \times \mathfrak{M}(F)$. The notation $\lambda \otimes \mu$ is justified as the topological vector space $\mathcal{K}(E \times F)$ is the completed tensor product of the topological vector spaces $\mathcal{K}(E)$, $\mathcal{K}(F)$ in one of the usual senses. Similarly for a finite number of factors. Infinite product measures and, more generally, projective limits of measures are introduced as follows. Let $\{E_i; i \in I\}$ be an infinite family of compact spaces. For each finite $J \subset I$, let μ_J be a positive measure on $E_J = \prod\{E_i; i \in J\}$. If $J \subset K \subset I$, J and K finite, assume that $\int f d\mu_J = \int (f \circ \mathrm{pr}_{JK}) d\mu_K$ for every $f \in \mathcal{C}(E_J)$, where $\mathrm{pr}_{JK}: E_K \rightarrow E_J$ is the natural projection. Then there is a unique positive measure μ on $E=\prod\{E_i; i \in I\}$ such that $\int f d\mu_J = \int (f \circ \mathrm{pr}_{JI}) d\mu$ for every $f \in \mathcal{C}(E_J)$ called the projective limit of the $\{\mu_J\}$. If μ_i is a positive measure on each E_i with $\int d\mu_i=1$ and μ_J is the finite product measure $\otimes\{\mu_i; i \in J\}$ on E_J, the projective limit so obtained is called the infinite product measure and denoted by $\otimes\{\mu_i; i \in I\}$.

Chap. IV (pp. 103–228) is the largest one. Let E be always a locally compact space, μ a positive measure on E. The author's procedure in going from continuous functions up to integrable ones consists in replacing Carathéodory's outer measure for sets by a similar use of upper integrals for functions, or, what amounts to the same, in doing with coordinate sets $\{(x, y); x \in E, 0 \leq y \leq f(x)\}$ to get the integral what is usually done with subsets of E to get the measure. Let $\mathcal{K}_+$ be the cone of all positive elements of $\mathcal{K}$ and $\mathcal{I}_+$ the cone of all positive lower-semicontinuous real functions on E. As the members of $\mathcal{I}_+$ are those functions expressible as suprema of members of $\mathcal{K}_+$, one sets

$$\mu^*(f)=\sup\{\mu(g); g \in \mathcal{K}, g \leq f\}.$$

For every positive real function f on E, one next defines $\mu^*(f)=\inf\{\mu^*(h); h \in \mathcal{I}_+, h \geq f\} = \int^* f(x) d\mu(x)$ as its upper integral. Every $A \subset E$ with characteristic function φ_A has an outer measure $\mu^*(A)=\mu^*(\varphi_A)$. The usual properties of outer measures, including those of the type of Fatou's lemma, are immediately given. A real function f on E is said to be μ-negligible if $\mu^*(|f|)=0$. Similarly for a set $A \subset E$ with $\mu^*(A)=0$. "Almost everywhere" means then outside a negligible set. Classes of μ-equivalent functions are introduced in the usual way. F being a real Banach space (with norm denoted as $|z|$ and not as $\|z\|$), for every function $\mathbf{f}: E \rightarrow F$ let $|\mathbf{f}|$ be the real function on E given by $x \rightarrow |\mathbf{f}(x)|$ and $N_p(\mathbf{f}) = (\int^* |\mathbf{f}|^p d\mu)^{1/p}$, $1 \leq p < \infty$. Call $\mathfrak{F}_F^p = \mathfrak{F}_F^p(E, \mu)$ the vector space of all functions $\mathbf{f}: E \rightarrow F$ such that $N_p(\mathbf{f})<\infty$. Here N_p is a semi-norm on $\mathfrak{F}_F^p$ under which this space is complete. As $\mathfrak{F}_F^p$ is inconveniently large, $\mathcal{L}_F^p$ is defined to be the closure in $\mathfrak{F}_F^p$ of its vector subspace $\mathcal{K}_F$. Finally L_F^p is the Banach space obtained from $\mathcal{L}_F^p$ by identifying equivalent functions. It is then readily verified that the vector subspace of all finite sums $\sum \mathbf{a}_k f_k$, where $\mathbf{a}_k \in F$ and $f_k \in \mathcal{K}$ is dense in $\mathcal{L}_F^p$ (a fact proved later for step functions on E with values in F too). Order properties in L^p (real functions) and Lebesgue's theorem for dominated convergence are proved. The integral of integrable functions $\mathbf{f} \in \mathcal{L}_F^1$ comes into consideration in the following way. As $\mathcal{K}_F$ is N_1-dense in $\mathcal{L}_F^1$ and the linear mapping $\mathcal{K}_F \rightarrow F$ given by $\mathbf{f} \rightarrow \int \mathbf{f} d\mu$ (weak vector-integral) is continuous from the N_1 semi-norm to the norm of F, it can be extended by continuity to a unique mapping $\mathcal{L}_F^1 \rightarrow F$, denoted again by $\mathbf{f} \rightarrow \int \mathbf{f} d\mu$. The usual properties of the integral and integrable functions are proved in reasonable detail. The notion of clan (Boolean ring of sets) is introduced in connection with approximation by step functions. The following general version of Riesz's theorem is given. Let Φ be a collection of subsets of a locally compact space E and $X \rightarrow \alpha X$ a finite positive real function on Φ. Assume that: $X, Y \in \Phi \rightarrow X \cup Y \in \Phi$, $X \cap Y \in \Phi$; if $K \subset E$ is compact, $U \subset E$ is open, $K \subset U$, there is an $X \in \Phi$ such that $K \subset X \subset U$; $X \subset Y \rightarrow \alpha X \leq \alpha Y$; $\alpha(X \cup Y) \leq \alpha X + \alpha Y$ and there is equality if $X \cap Y = \varnothing$. Then there is a positive measure μ on E such that every $X \in \Phi$ is μ-integrable (i.e., has a μ-integrable characteristic function) and $\alpha X = \mu X$ if and only if α is regular, i.e., for every $\epsilon>0$ and $X \in \Phi$ there are $K \subset X$ compact and $U \supset X$ open such that $|\alpha Y - \alpha X| < \epsilon$ for $K \subset Y \subset U$; and μ is unique. Using Lusin's classical theorem as motivation, a measurable function f is defined as one from E into a topological space F such that, for every compact $K \subset E$, there is a μ-negligible $N \subset K$ and a partition of $K \cap \complement N$ into a sequence of compact sets K_n such that the restrictions of f to each K_n are continuous. Local negligibility and several criteria for measurability are considered, most of them being of a familiar type except for the following condition used in place of the classical condition of countable base for F: for every compact $K \subset E$, there is a countable subset $H \subset F$ such that $f(x) \in \bar{H}$ for almost all $x \in K$. Egoroff's theorem is also proved. The chapter ends with a section in which the convexity inequalities for $\mathcal{L}_F^p$ are treated with greater detail. The space $\mathcal{L}_F^\infty$ of all functions $\mathbf{f}: E \rightarrow F$ (F Banach space) measurable and bounded in measure with semi-norm $N^\infty(\mathbf{f})$ given by the maximum in measure and the associated L_F^∞ are introduced so that the properties can be proved for $1 \leq p \leq \infty$ (and not only $<\infty$). Chap. IV includes a few sections in small print devoted to abstract measures. These are introduced as follows. Let A be a set and $\mathfrak{R}$ a vector subspace of the space of all finite real functions on A such that $|f| \in \mathfrak{R}$ whenever $f \in \mathfrak{R}$. A

linear positive functional μ on $\mathfrak{R}$ is called an abstract measure if it satisfies the following condition: if an increasing sequence $(f_n)\subset\mathfrak{R}$ has $\sup f_n \in \mathfrak{R}$, then $\mu(\sup f_n)=\sup \mu(f_n)$. An upper integral $\mu^{**}(f)$ is then constructed for every positive real function f on A and most of the results given in the main text are shown or said to remain valid in the abstract case [cf. M. H. Stone, Proc. Nat. Acad. Sci. U. S. A. **34**, 336–342, 447–455, 483–490 (1948); **35**, 50–58 (1949); these Rev. **10**, 24, 107, 239, 360].

Only a few trivial misprints were observed. For instance, one should read $(M((f+g)^p))^{1/p}$ on p. 12, l. 9; $\mu(f)=\nu(f)$ on p. 46, l. 7; $\Pi_{i\in I}E_i$ on p. 99, l. 15b; $\varphi((\bar{f}_n))$ on p. 121, l. 9.

L. Nachbin (Rio de Janeiro).

Edwards, R. E. A theory of Radon measures on locally compact spaces. Acta Math. **89**, 133–164 (1953).

This is a very readable treatment of the theory of positive linear functionals on various spaces of real-valued continuous functions defined on locally compact Hausdorff spaces. It can be recommended as a good introduction for the general mathematical reader to this special branch of functional analysis. Let X be a locally compact Hausdorff space, $\mathfrak{C}$ the set of continuous real-valued functions on X each having compact support, and μ an arbitrary positive linear functional on $\mathfrak{C}$ (differences of such functionals are called, by the usual abuse of language, Radon measures on X). Let B be the σ-ring generated by the compact subsets of X, $\mathbf{B}^*$ the σ-algebra generated by the closed subsets of X. About two-thirds of the paper is concerned with representation of μ as an integral, $\mu(f)=\int_X f(x)d\mu(x)$, where the second μ is a countably additive, non-negative measure defined on some σ-ring containing **B**. This matter has of course been treated by other writers, e.g., P. R. Halmos [Measure theory, Van Nostrand, New York, 1950, Ch. X; these Rev. **11**, 504]; N. Bourbaki [the book reviewed above]; the reviewer and H. S. Zuckerman [Nagoya Math. J. **3**, 7–22 (1951); these Rev. **14**, 362]. The present exposition is based on an extension μ_0 of μ of the Daniell type. Some novelties appear in the manner of this extension. The measure function μ associated with the functional μ is defined by $\mu(E)=\sup \mu_0(\chi_{E\cap K})$, where K runs through the family of compact subsets of X and E is any set such that $\chi_{E\cap K}$ is summable with respect to μ_0 for all compact sets K. The family **F** of such sets E is a σ-algebra containing $\mathbf{B}^*$, $(X, \mathbf{F}, \mu)$ is a locally bounded measure space, and the integral $\int_X f(x)d\mu(x)$ represents the functional μ on $\mathfrak{C}$. It is shown that at least the sets in **B** are regular. An outer measure μ_e is defined by $\mu_e(A)=\inf \mu(G)$, taken over all open sets $F\supset A$, for all $A\subset X$. Connections between the sets Carathéodory measurable with respect to μ_e and the family **F** are explored. A short and elegant proof of Fubini's theorem, due in part to J. L. B. Cooper, is given. The final part of the paper deals with various ways of topologizing **C** as a locally convex linear topological space so that the dual of **C** is exactly the set **M** of all Radon measures. The weakest such topology is clearly the weak topology induced on **C** by **M**, and the strongest is constructed here explicitly, in a manner suggested by the $(\mathfrak{LF})$-topologies of J. Dieudonné and L. Schwartz [Ann. Inst. Fourier Grenoble **1**, 61–101 (1950); these Rev. **12**, 417]. The paper closes with some observations on Haar measure and the construction of duals of certain function spaces.

E. Hewitt (Seattle, Wash.).

Aumann, Georg. Integralerweiterungen mittels Normen. Arch. Math. **3**, 441–450 (1952).

This paper involves a modification of the development of M. H. Stone of a Lebesgue type of integral on a general set [Proc. Nat. Acad. Sci. U. S. A. **34**, 336–342 (1948); these Rev. **10**, 24]. Following Stone, there is postulated on the general set A, a class $\mathfrak{T}$ of elementary functions $t(x)$ which form a linear lattice and an elementary integral $T(t)$ linear positive on $\mathfrak{T}$. Instead, however, of defining a norm on the set $\mathfrak{G}$ of all functions on A in terms of the elementary integral, a pseudo-norm N is postulated which satisfies also the monotone condition: if $0\leqq g_1(x)\leqq g_2(x)$, then $Ng_1\leqq Ng_2$. The integral $T(t)$ is assumed to be uniformly continuous on $\mathfrak{T}$ relative to N. Then the closure of the class $\mathfrak{T}$ agrees with the class $\mathfrak{T}^*$ of functions on which there exists a finite extended integral. In the obvious way, by the introduction of equivalence classes modulo N, it is possible to set up the class $\tilde{\mathfrak{T}}$ of function sets and integrals which is a complete normed vector space. The following three cases are considered: (a) uniform norm: $Ng=\sup |g(x)|$ for x on A; (b) Riemannian norm: $Ng=\inf T(t)$ for $t(x)\geqq|g(x)|$; (c) Lebesgue norm: $Ng=\inf \sum_n T(t_n)$ for $t_n\geqq 0$ and $\sum_n t_n(x)\geqq g(x)$. The last named is identical with the Stone procedure and requires the usual sum postulate on $T(t)$ to insure the continuity property of T. There is consideration of the special case where the class $\mathfrak{T}$ is a set of step functions, and in particular when A is a linear interval.

T. H. Hildebrandt (Ann Arbor, Mich.).

Brownell, F. H. Translation invariant measure over separable Hilbert space and other translation spaces. Pacific J. Math. **2**, 531–553 (1952).

The author considers the real Hilbert space l_2 of sequences $x=\{x_n\}$ for which $\sum_1^\infty x_n^2<+\infty$. As is well known, it is not possible to define a non-trivial translation-invariant measure in l_2. The author restricts attention to measures defined in certain subsets of l_2. In a set X defined by inequalities $|x_n|\leqq h(n)$ for all n, where $\sum_{N+1}^\infty h(n)^2<+\infty$ for some N, the product measure of the linear measures on the intervals $|x_n|\leqq h(n)$ (normalized by the factor $1/2h(n)$ for $n\geqq N+1$) is a translation-invariant measure. In a set Y defined by inequalities $\sum_n^\infty x_i^2\leqq f(n)$ for all n, where $f(n)\downarrow 0$, a translation-invariant measure is obtained by an adaptation of Haar's method.

B. Jessen (Copenhagen).

Pfeiffer, Paul E. Equivalence of totally finite measures on infinite product spaces. Ann. of Math. (2) **56**, 520–536 (1952).

The object of this paper is to study conditions under which two probability measures defined on a countably infinite product space are equivalent, i.e., have the same null-sets. This problem was treated for the case of product measures by Kakutani [Ann. of Math. (2) **49**, 214–224 (1948); these Rev. **9**, 340], and for the case of arbitrary measures by Kawada [Math. Japonicae **1**, 170–177 (1949); these Rev. **11**, 89]. For the case of arbitrary measures the author obtains a new criterion, which in the case of product measures leads to a very simple result stating that two such measures are equivalent if and only if their contractions to certain sub-algebras of sets are equivalent. New proofs of Kawada's and Kabutani's results are obtained.

B. Jessen (Copenhagen).

Hewitt, Edwin. A problem concerning finitely additive measures. Mat. Tidsskr. B. **1951**, 81–94 (1951).

Let E_I denote the smallest algebra of sets which contains all intervals $[a, b)$, $0\leq a<b\leq 1$, and B (L) the σ-algebra of Borel (Lebesgue) sets. A finitely additive measure $\varphi\geq 0$ on E_I is said to be purely finitely additive if $0\leq\psi\leq\phi$ for a countably additive ψ implies that $\psi=0$. Let λ_t $(0<t\leq 1)$ be defined for all $A\in E_I$ as follows: $\lambda_t(A)=1$ if $[t-\delta, t)\subset A$ for some $\delta>0$, $\lambda_t(A)=0$ otherwise. Then the measure λ_t are the only purely additive measures on E_I assuming the values 0 and 1, and any bounded finitely additive measure on E_I is of the form $\phi=\sum_1^\infty \alpha_n\lambda_{t_n}+\psi$, where $\sum|\alpha_n|<\infty$ and ψ is countably additive. The author describes all the finitely additive extensions of ϕ over B and over L, and defines integrals with respect to such measures ϕ. These integrals permit one to describe the bounded linear functionals and the Gelfand representation of the Banach algebra $P=\{f\}$ of all the uniform limits of linear combinations of characteristic functions of sets $[a, b)$ with the norm $\|f\|=\sup_{0\leq t<1}|f(t)|$.

M. Cotlar (Buenos Aires).

***Rogosinski, Werner W. Volume and integral.** Oliver and Boyd, Edinburgh and London; Interscience Publishers, Inc., New York, 1952. x+160 pp. $1.75.

This book has been written "primarily for third year Honours students" in the Universities of Great Britain. The first chapter deals with sets and sequences of sets in n-dimensional Euclidean space **E**, and their properties. An adequate foundation is laid on which to build the theories of the Riemann and Lebesgue integrals. The next two chapters give the metric properties of sets relative to the Peano-Jordan content and Lebesgue measure. Content and measure are considered as special cases of the more general idea of volume $V(E)$, E a set in **E**. The volume $V(E)$ is required to satisfy the following postulates: I) $V(E)=0$; II) $V(E_1+\cdots+E_m)=V(E_1)+\cdots+V(E_m)$; III) for an elementary set P (interval, polyhedra, etc.) $V(P)$ has its elementary value; IV) congruent sets have the same value. Content is defined and shown to satisfy these postulates. The deficiencies of content are discussed, e.g., it is shown that the set of points with rational coordinates does not have content. In Chapter III the idea of Lebesgue measure is covered in the usual way. An example of a non-measurable set is given, and the Vitali covering theorem is proved.

The second part of the book deals with the integral concept. First the Riemann integral of a function on an n-dimensional Euclidean set which has content is defined as the content of the ordinate set in $(n+1)$-dimensional space, the Lebesgue integral of such a function on a measurable set as the $(n+1)$-dimensional measure of the ordinate set. The properties of these integrals are given and the relations between them are shown. The bounded convergence theorem, Egoroff's theorem, and Fubini's theorem are proved. The final Chapter VI deals with differentiation and integration. The properties of the indefinite integral, including its absolute continuity, are studied as are the properties of functions of bounded variation including the proof that such functions have a finite derivative almost everywhere which is summable.

This book is well suited for the purpose the author had in mind. It gives a complete treatment of absolutely convergent integrals in Euclidean space. The proofs are brief and elegant, and at the same time easy to follow.

R. L. Jeffery (Kingston, Ont.).

Pregnolato, Giuseppe T. Media integrale e integrale di Lebesgue. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15**(**84**), 498–508 (1951).

The paper gives a definition of the Lebesgue integral mean $\int_a^b f(x)dx/(b-a)$ for $f(x)$ bounded, a and b finite, by a mean value process. The basic lemma is: If A_0 and B_0 are two linear sets with $\mu A_0\leq\mu B_0$ (μ being Lebesgue measure), then there exists a set A_ω of zero measure and a one-to-one mapping τ of A_0-A_ω on a subset of B_0 such that if E is any measurable subset of A_0-A_ω, then E and τE have the same measure. If now $f(x)$ is a measurable and bounded function with bounds m and M on the linear interval $T=(a, b)$, set

$$A_0=E[m\leq f(x)\leq (m+M)/2]$$

and

$$B_0=E[(m+M)/2<f(x)\leq M].$$

Assume $\mu A_0\leq\mu B_0$ and apply the lemma to define the function $f_1(x)=[f(x)+f(\tau x)]/2$ on A_0-A_ω, $=[f(x)+f(\tau^{-1}x)]/2$ on $\tau(A_0-A_\omega)$, and $=f(x)$ elsewhere. Then if $T_1=T-A_\omega$, we have $\int_T f(x)=\int_{T_1}f_1(x)$ and the range of $f_1(x)$ on T_1 is at most of length $\frac{3}{4}(M-m)$. Repeating the process with $f_1(x)$ yields a sequence of functions converging to a constant function on a set T_ω which differs from T by at most a set of zero measure. This constant value is the Lebesgue integral mean.

T. H. Hildebrandt (Ann Arbor, Mich.).

Hadwiger, H. Translationsinvariante, additive und schwachstetige Polyederfunktionale. Arch. Math. **3**, 387–394 (1952).

The author considers functionals $\varphi=\varphi(A)$ defined for all polyhedra A in k-dimensional Euclidean space, which are translation-invariant, additive, and weakly continuous, i.e., are continuous by variations consisting only in translations of the faces. A dilation, by which A is replaced by the homothetic polyhedron λA $(\lambda>0)$, is of this type. If $\varphi(\lambda A)=\lambda^i\varphi(A)$, the functional is called homogeneous of degree i. The author gives a complete characterization of all functionals φ. The explicit representation, obtained by recursion with respect to k, shows in particular that every φ is a sum of homogeneous functionals of degrees $i=1, 2, \cdots, k$. It also shows that a homogeneous functional of degree i satisfies $\varphi(\bar{A})=(-1)^{k-i}\varphi(A)$, where $\bar{A}$ is symmetric to A with respect to a point.

B. Jessen.

Hadwiger, H. Lineare additive Polyederfunktionale und Zerlegungsgleichheit. Math. Z. **58**, 4–14 (1953).

Two polyhedra A and B of k-dimensional Euclidean space R_k are called "G-zerlegungsgleich", where G is a group of movements in R_k containing the translation group (notation $A\sim B$) if $A=\sum_1^n A_\nu$ and $B=\sum_1^n B_\nu$, where A_ν and B_ν are polyhedra such that A_ν may be carried into B_ν by a movement of G. They are called "G-zerlegungsgleich mod Z" if there exist cylinders U and V such that $A+U\sim B+V$. A cylinder in R_k is defined as the Minkowski sum of polyhedra in spaces of lower dimensions spanning R_k. In continuation of previous investigations [esp. Comment. Math. Helv. **24**, 204–218 (1950); and Math. Z. **55**, 292–298 (1952); these Rev. **12**, 526; **14**, 309] the author proves that two polyhedra A and B are G-zerlegungsgleich mod Z if and only if $\varphi(A)=\varphi(B)$ for every linear, additive polyhedron functional φ which is invariant under the movements of G. In spaces of even dimension, two arbitrary polyhedra are G-zerlegungsgleich mod Z, when G is the group consisting of all translations and all symmetries with respect to points of R_k.

B. Jessen (Copenhagen).

Rogers, C. A. Certain integrals over convex sets. J. London Math. Soc. **28**, 293–297 (1953).

The author proves the following theorem. Let $\rho(r)$ be a real-valued function on the positive reals which is non-decreasing for $0<r<1$ and which vanishes for $r>1$. In euclidean n-space E_n let K be a closed convex set such that $\mu(K)<\mu(S)$, μ denoting Lebesgue measure and S the closed solid unit sphere about the origin 0 in E_n. Then there is a segment $S(K)$ of S with $\mu(S(K))=\mu(K)$ such that $\int_K\rho(|X|)d\mu\leqq\int_{S(K)}\rho(|X|)d\mu$, where $|X|$ denotes the distance of the point X of E_n from 0. *T. A. Botts.*

Hsu, L. C. A theorem concerning an asymptotic integration. Chung Kuo K'o Hsüeh (Chinese Science) **2**, 149–155 (1951). (Chinese. English summary)

Let g and f be real bounded functions Lebesgue integrable over a measurable set E and suppose that (1) f attains a positive absolute maximum at an interior point u such that $\lim_{x\to u}|f(x)-f(u)|/|x-u|^c=k>0$ where $c>0$; (2) E has density 1 at u; (3) u belongs to the Lebesgue set for g and $g(u)\neq0$. Then for every $a\geqq0$ we have, as $n\to\infty$,

$$\int_E g(x)|x-u|^a(f(x))^n dx\sim 2c^{-1}(nk)^{-(a+1)/c}$$
$$=\Gamma\left(\frac{a+1}{c}\right)g(u)(f(u))^{n+(a+1)/c}.$$

This result includes Laplaces's theorem, B. Levi's extension [Publ. Inst. Mat. Univ. Nac. Litoral **6**, 341–351 (1946); these Rev. **8**, 16], and a theorem of Widder [The Laplace transform, Princeton, 1941, pp. 298–299; these Rev. **3**, 232]. *K. L. Chung* (Syracuse, N. Y.).

Theory of Functions of Complex Variables

Marković D. Sur les zéros réels des dérivées des quelques fonctions. Bull. Soc. Math. Phys. Serbie **4**, no. 3–4, 1–5 (1952). (Serbo-Croatian summary)

By application of Leibniz' rule for the derivative of a product of two functions, the author finds a number of theorems on the real zeros of the derivatives of a polynomial $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ $(a_n>0)$, a rational function $P(x)/Q(x)$, or the function $\exp(P(x))$. For example, let c_j denote the real zeros of $P(x)$, α_k the real part of its complex zeros, $a=\max(c_j,\alpha_k)$, $b=\min(c_j,\alpha_k)$, and let $P^{(m)}$ denote the mth derivative of $P(x)$. Then, unless identically zero, the derivatives of $P(x)$ satisfy the inequalities $P^{(m)}<0$ for $x>a$; but $(-1)^rP^{(2m)}>0$ and $(-1)^rP^{(2m+1)}<0$ for $x<b$, where r is the number of real zeros of P. *M. Marden* (Milwaukee, Wis.).

Kjellberg, Bo. On integral functions bounded on a given set. Mat. Tidsskr. B. **1952**, 92–99 (1952).

The author proves some theorems of the general form that an entire function $f(z)$ of slow growth, satisfying $|f(z)|\leqq1$ on a large enough set, is constant. Let $|f(z)|\leqq1$ at the points z_n and let r_n be given positive numbers. The author first proves a theorem, analogous to Jensen's formula, which gives an upper estimate for the number of circles of center z_n and radius $2r_n$ in which $\max|f(z)|>2$. As a first application he proves that if $f(z)$ is at most of order ρ, minimal type, and is bounded on $\{z_n\}$, where every circle $|z|=r_0^k$ $(r_0>1;\ k=1,2,\cdots)$ is covered by the system of circles $|z-z_n|\leqq A|z_n|^{1-\rho}$, then $f(z)$ is constant. Then he uses a theorem of Pfluger [C. R. Acad. Sci. Paris **229**, 542–543 (1949); these Rev. **11**, 94] to obtain an upper estimate for $\log M(r)$. Using this he proves that if $\rho\geqq\frac12$ and $f(z)$ is at most of order ρ, convergence class, and bounded on a set $\{z_n\}$, it is constant provided that every sector $\theta\leqq\arg z\leqq\theta+\pi/\rho$ contains a subset $\{z_m'\}$ for which

$$\sup_m\{|z'_{m+1}|^\rho-|z_m'|^\rho\}\leqq A,$$

where A is independent of θ. There is a similar result for $\rho<\frac12$. *R. P. Boas, Jr.* (Evanston, Ill.).

Hayman, W. K. An integral function with a defective value that is neither asymptotic nor invariant under change of origin. J. London Math. Soc. **28**, 369–376 (1953).

For meromorphic functions it is known that a defective value is not always asymptotic [Mme. Laurent Schwartz, C. R. Acad. Sci. Paris **212**, 382–384 (1941); these Rev. **2**, 357], and that the defects are sometimes altered by a change of origin [Dugué, ibid. **225**, 555–556 (1947); these Rev. **9**, 139]. The author shows that the entire function

$$f(z)=\prod_{n=1}^\infty\left\{1+\left(\frac{z}{n}\right)^{3n}\right\}^{2^n}$$

has the following properties. (i) If $\mu(r,f)$ and $M(r,f)$ denote, respectively, the minimum and the maximum of $|f|$ on $|z|=r$, then

$$\mu(n+\tfrac12,f(z))>\{M(n+\tfrac12,f(z))\}^{4/5}\quad(n>n_0).$$

It follows that $f(z)$ has no finite defective value and that $f(a+z)$ has no finite asymptotic value for any a. (ii) For any a with $|a|\geqq\frac12$, zero is a defective value of $f(a+z)$ with defect $\delta(0)\geqq10^{-21}/|a|$, and

$$\mu(r,f(a+z))<\{M(r,f(a+z))\}^{-\frac{1}{40}\log\log\log M(r,f(a+z))}\qquad(r>r_0).$$

It is stated that one can construct entire functions of (sufficiently large) finite order which have a defective value but no asymptotic value. It is not known whether the defects of a function of finite order may be altered by a change of origin. [The paper contains several misprints. For Pflüger read Pfluger.] *J. Korevaar* (Madison, Wis.).

Buck, R. Creighton. On admissibility of sequences and a theorem of Pólya. Comment. Math. Helv. **27**, 75–80 (1953).

Let K be the class of entire functions of order one, mean type satisfying $|f(iy)|=O(e^{c|y|})$ for a $c<\pi$. The sequence b_n is said to be admissible T_n^* for K if there is an $f(z)\in K$ such that $T_n^*(f)=\Delta^nf(0)=\sum_{k=0}^n\binom{n}{k}(-1)^{n-k}f(k)=b_n$ $(n=0,1,2,\cdots)$. The author proves that: 1) b_n is admissible T_n^* if and only if $g(z)=\sum b_nz^n$ is analytic at zero and can be continued analytically to the interval $-1\leqq x\leqq0$; 2) $g(z)$ is entire if and only if $f(z)$ is of zero type. Combining these facts with known results about Newton series he proves: 3) If $\limsup_{n\to\infty}|a_n|^{1/n}\leqq1$ and $\Delta^na_0=0$ for all n of a set of integers of density d, then, for $d>1/3$, $\sum_{n=0}^\infty\binom{z}{n}\Delta^na_0$ converges to an $f(z)\in K$ of type not exceeding

$$\max(0,\log(1+2\cos\pi d)).$$

The example of $\sum\Delta^na_0z^n=(z-1)/(z+1)(z^3-1)$ shows that the lower bound 1/3 is exact. d may be 1 for a non-polynomial $f(z)$, e.g.,

$$f(z)=\sum_{n=1}^\infty\binom{z}{n^2}\frac{1}{n!}.$$

This is in contrast to a theorem of Agnew and the reviewer [Agnew, Amer. J. Math. 66, 339–340 (1944); these Rev. 6, 46] that $a_n = O(1)$, $d > 1/2$ implies $f(z) \equiv$ constant. The proofs are based on the formulae

$$g(z) = \sum \Delta^n f(0) z^n = \frac{1}{2\pi i} \int_\Gamma \varphi(w)[1-(e^w-1)z]^{-1} dw,$$

$$f(z) = \frac{1}{2\pi i} \int_C t^{-1} g(t)[(1+t)/t]^z dt,$$

where φ is the Borel-transform of $f(z)$, Γ encircles the conjugate indicator diagram of $f(z)$, and C encircles the line segment $-1 \leqq t \leqq 0$. For a function $f(z)$ of zero type the first formula gives after expansion

$$g(z) = \sum_0^\infty f(n)[z/(1+z)]^{n-1} \quad (|z| < |1+z|)$$

or

$$g(z) = (-1/z) \sum_{n=0}^\infty f(-n-1)[(1+z)/z]^n \quad (|1+z| < |z|).$$

This leads to a very short proof of Pólya's theorem: If $f(z)$ is of order one, zero type, and $|f(n)| < K$, then $f(z) \equiv$ constant. For without loss of generality we may assume that $\sum |f(n)| < \infty$ and then the formulae just quoted show that $g(z)$ is bounded and hence a constant. The same is therefore true of $f(z)$. *W. H. J. Fuchs.*

Buck, R. Creighton. Essentially admissible sequences. Proc. Amer. Math. Soc. 4, 387–390 (1953).

Let X denote the set of all complex sequences $\alpha = \{a_n\}$ $(n = 0, 1, \cdots)$ such that $\|\alpha\| = \sup_n |a_n|^{1/(n+1)} < \infty$. A sequence $\alpha \in X$ is called admissible (essentially admissible) if there is an entire function $f(z)$ of exponential type which satisfies $f(iy) = O\{\exp(c|y|)\}$ for some $c < \pi$, and which is such that $f(n) = a_n$ for all $n \geqq 0$ (for all sufficiently large n). Unlike admissibility, essential admissibility is a property which is preserved if one alters a sequence in a finite number of places. Define the distance between α and β of X by $\|\alpha - \beta\|$, and let A denote the set of all essentially admissible sequences α. The author shows that A is of first category in X, and not dense in X. The proofs depend on the author's previous characterizations of admissible sequences [Duke Math. J. 13, 541–559 (1946); these Rev. 8, 371; see also the paper reviewed above], in particular the result that a sequence α is admissible if and only if $g(z) = \sum (\Delta^n a_0) z^n$ is regular in a neighborhood of the interval $-1 \leqq x \leqq 0$.

J. Korevaar (Madison, Wis.).

Parthasarathy, M. A theorem on integral functions. J. London Math. Soc. 28, 377–379 (1953).

Let $f(z)$ be a transcendental entire function, and let $\mu(r)$ be a majorant of $M(r, f)$ for $r_0 < r < \infty$ which possesses a continuous nth derivative. Hayman conjectured (and proved for $n = 1, 2$) [Proc. Cambridge Philos. Soc. 48, 93–105 (1952); these Rev. 13, 631] that there is a sequence of values of r tending to infinity on which $M(r, f^{(n)}) = O\{\mu^{(n)}(r)\}$. [The author states that Hayman's conjecture has been proved by F. M. Stewart.] The author now makes the additional assumptions that $\mu^{(n)}(r)$ is positive and increasing for $r > r_0$ and that $\mu'(r)/\mu(r) \to \infty$ as $r \to \infty$. Then for every $\epsilon > 0$, $M(r, f^{(n)}) = o\{\mu^{(n)}(r+\epsilon)\}$ as r tends to infinity continuously.

J. Korevaar (Madison, Wis.).

Timan, A. F. On interference phenomena in the behavior of entire functions of finite degree. Doklady Akad. Nauk SSSR (N.S.) 89, 17–20 (1953). (Russian)

Let $B_\sigma^{(q)}$ denote the class of entire functions which are of exponential type σ and $o(|x|^q)$ on the real axis, with $\{f(k\pi/\sigma)\}$ bounded; B_σ is the subclass whose members are bounded on the real axis. S. Bernstein [Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 421–444 (1948); these Rev. 10, 363] established the "interference theorem" that if $f(z)$ belongs to $B_\sigma^{(1)}$, then $f(z+\frac{1}{2}\pi/\sigma) + f(z-\frac{1}{2}\pi/\sigma)$ belongs to B_σ (with an explicit bound). The author generalizes this by obtaining a condition on the function $\rho(x)$, of bounded variation, satisfying $\int_{-\infty}^\infty e^{\tau|x|} |d\rho(x)| < \infty$ (some $\tau > \sigma$), which is necessary and sufficient for the function $\int_{-\infty}^\infty f(z+t) d\rho(t)$ to belong to B_σ when $f(z)$ belongs to $B_\sigma^{(q)}$. The condition is that for $k = 0, 1, \cdots, [q-2]$, the integrals $\int_{-\infty}^\infty t^k (\sin, \cos) \sigma t d\rho(t)$ should be zero. [Since a function of $B_\sigma^{(q)}$, $q > 1$, is necessarily of the form $g(z) + P(z) \sin \sigma z$, with g in $B_\sigma^{(1)}$ and P a polynomial, the case $q > 1$ is readily deduced from the case $q = 1$.] In particular, $f(z+\lambda) + f(z-\lambda)$ has the property in question $(q = 1)$ only when $\lambda = \frac{1}{2} m\pi/\sigma$; the author determines the asymptotic behavior (as $m \to \infty$) of the bound in this case.

R. P. Boas, Jr. (Evanston, Ill.).

Klimczak, W. J. Differential operators of infinite order. Duke Math. J. 20, 295–319 (1953).

The paper is concerned with the theory of differential operators of infinite order which are of the form $G(D) = \sum g_k D^k$, where $G(w)$ is an entire function of w and D is a linear differential operator of finite order s having coefficients analytic in a domain Δ. The author generalizes some earlier work due to E. Hille [same J. 5, 875–936 (1939); 7, 458–495 (1940); these Rev. 1, 141; 2, 184]. The operator $G(D)$ is said to be uniformly applicable to the class $F(\Delta_1)$ of functions $\{f(z)\}$ analytic in a domain $\Delta_1 \subset \Delta$ if the series $G(D)f(z) = \sum g_k D^k f(z)$ converges uniformly in each compact subset of Δ_1 for each $f \in F(\Delta_1)$. It is shown that $G(D)$ is uniformly applicable to all classes $F(\Delta_1)$ if and only if $G(w)$ is an entire function of order $\sigma \leqq 1/s$ and of minimal type if $\sigma = 1/s$. If the coefficients of D are polynomials in z and if the coefficient of the highest derivative in D is at most of order $s-1$, then $G(D)f(z)$ will be an entire function provided the orders of f and G satisfy a certain inequality determined by an order relation which depends on the degrees of the polynomial coefficients in D. Finally for the case $s = 2$, the author obtains results of this type which are sharp; that is, necessary and sufficient conditions are given.

R. S. Phillips (New Haven, Conn.).

Kaplan, Wilfred. Close-to-convex schlicht functions. Michigan Math. J. 1 (1952), 169–185 (1953).

Let $f(z)$ be analytic, $f'(z) \neq 0$ in $|z| < 1$. The following three conditions are shown to be equivalent. a) There exists a function $\phi(z)$ schlicht and convex in $|z| < 1$ such that $\operatorname{Re}\{f'(z)/\phi'(z)\} > 0$. b) There exists a function $w = \phi(z)$ mapping $|z| < 1$ onto a convex domain D and a function $g(w)$ such that $\operatorname{Re}\{g'(w)\} > 0$ in D and $f(z) = g\{\phi(z)\}$. c) Let γ_r be the curve $w = f(re^{i\theta})$, $0 \leqq \theta \leqq 2\pi$, and let $P(r, \theta) = \theta + \arg f'(re^{i\theta})$ be the angle which the tangent to γ_r at $f(re^{i\theta})$ makes with the imaginary w-axis. Then we have for some r arbitrarily close to one, and hence for $0 < r < 1$, $P(r, \theta_2) > P(r, \theta_1) - \pi$ whenever $\theta_1 < \theta_2 \leqq \theta_1 + 2\pi$.

The author calls the corresponding functions $f(z)$ close-to-convex. Such functions are necessarily schlicht. Among close-to-convex functions are those mapping the origin onto a star-shaped domain (with respect to an arbitrary centre)

or onto a domain which cuts each of a system of parallel lines in at most a single segment; further, functions representable as

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}h(\theta)d\theta$$

where $h(\theta)$ is non-decreasing for $0\leqq\theta\leqq\alpha$ and non-increasing for $\alpha\leqq\theta\leqq 2\pi$. *W. K. Hayman* (Exeter).

Jenkins, James A. On values omitted by univalent functions. Amer. J. Math. **75**, 406–408 (1953).

Let $f(z)=z+a_2z^2+\cdots$ be regular and schlicht for $|z|<1$. Let $L(f,r)$ be the length of the set of values on $|w|=r$ not covered by the image D_f of $f(z)$. If $r\leqq 1/4$, $L(f,r)=0$. If $r\geqq 1$, $L(f,r)$ may take any value between 0 and $2\pi r$. If $1/4<r<1$, it follows from subordination and Pólya's theory of circular symmetrisation [C. R. Acad. Sci. Paris **230**, 25–27 (1950); these Rev. **11**, 435], that $L(f,r)$ is maximal when D_f is bounded by an arc on $|w|=r$ symmetrical about the positive real axis and the part $w\geqq r$ of this axis. For this map the author shows that $L(f,r)=2r\cos^{-1}(8r^{\frac{1}{2}}-8r-1)$. By integrating this bound from $r=1/4$ to $r=1$ he obtains the bound $A_f\geqq .5387\pi$, where A_f is the area of $\{D_f\cap|w|<1\}$, thus sharpening the bound $A_f\geqq\frac{1}{2}\pi$ due to Goodman [Bull. Amer. Math. Soc. **55**, 363–369 (1949); these Rev. **10**, 601]. *W. K. Hayman* (Exeter).

Bochner, S. Connection between functional equations and modular relations, and functions of exponential type. J. Indian Math. Soc. (N.S.) **16**, 99–102 (1952).

In a former paper [Ann. of Math. (2) **53**, 332–363 (1951); these Rev. **13**, 920] the author has introduced residual functions as follows: $P(x)$ is residual if (i) it is defined and differentiable in $0<x<\infty$, and, for some $\gamma>0$, $P(x)=0(x^{-\gamma})$ as $x\to 0$ and $P(x)=0(x^{\gamma})$ as $x\to\infty$; (ii) the two functions $\chi_r(s)=\int_0^1P(x)x^{s-1}dx$ and $\chi_l(s)=\int_1^\infty P(x)x^{s-1}dx$ (the first is defined in a right half-plane, the second in a left half-plane) can be continued into one another in a domain D, which is the exterior of a bounded closed set S; and (iii) if for the joint value of the continued function $\chi_0(s)$

$$\lim_{|t|\to 0}\chi_0(\sigma+it)=0\quad\text{uniformly in every finite interval }\sigma_1\leqq\sigma\leqq\sigma_2. \tag{1}$$

The role of the residual function is as follows. If the Mellin integrals $\chi_1(s)=\int_0^\infty x^{s-1}\phi(x)dx$, $\chi_2(s)=\int_0^\infty x^{s-1}\psi(x)dx$ satisfy the functional equation

$$\chi_1(s)=\chi_2(\delta-s)\quad\text{for some }\delta>0, \tag{2}$$

then they satisfy the modular relation

$$\phi(x)-x^{-\delta}\phi\left(\frac{1}{x}\right)=P(x) \tag{3}$$

for some suitable residual function $P(x)$.

In this paper the author gives an alternate equivalent description. $P(x)$, defined in $0<x<\infty$, is residual if and only if it can be represented as a series $P(x)=\sum_0^\infty(c_n/n!)(\log 1/x)^n$ with $\gamma_0=\limsup_{n\to\infty}|c_n^{1/n}|<\infty$ (so $f(y)=P(e^{-y})$ is of exponential type in y) and $\chi_0(s)$ (see (1)) is the Borel-transform $\chi_0(s)=\sum_0^\infty c_n/s^{n+1}$. Also, if $P(x)$ appears in a modular relation (3), and if $\chi(s)$ is the joint value of the continuations of $\chi_1(s)$ and $\chi_2(\delta-s)$ in a domain D as before (see (2)), then $\chi_0(s)$ is the singular part of $\chi(s)$ in the sense that $\chi(s)-\chi_0(s)$ is an entire function in s. *S. C. van Veen.*

Havinson, S. Ya. On extremal properties of functions mapping a region on a multi-sheeted circle. Doklady Akad. Nauk SSSR (N.S.) **88**, 957–959 (1953). (Russian)

The author gives a number of theorems on extremal problems and their extremal functions, generalizing various classical results as well as more recent work of Goluzin [Mat. Sbornik N.S. **18(60)**, 213–226 (1946); these Rev. **8**, 22], Ahlfors [Duke Math. J. **14**, 1–11 (1947); these Rev. **9**, 24], and Garabedian [Trans. Amer. Math. Soc. **67**, 1–35 (1949); these Rev. **11**, 340]. He considers a bounded region G, whose boundary Γ consists of n rectifiable curves, and the following classes of functions analytic in G: C_A, functions continuous in $\bar{G}$; C_A', the subclass of C_A for which $|f(z)|\leqq 1$; B, bounded functions; B', functions with $|f(z)|\leqq 1$; E_1, functions which are the Cauchy integrals of their boundary values. Then for any $\omega(x)$ which is summable on Γ, the supremum over C_A' or B' of $|\int_\Gamma f(x)\omega(x)dx|$ is equal to the infimum for φ in E_1 of $\int_\Gamma|\omega(x)-\varphi(s)|ds$. Extremal functions $f^*\in B'$ and $\varphi^*\in E_1$ exist, and are characterized by the fact that, almost everywhere on Γ,

$$f^*(x)[\omega(x)-\varphi^*(x)]dx=e^{i\alpha}|\omega(x)-\varphi^*(x)|ds.$$

If Γ consists of analytic curves, and $\omega(x)$ is analytic on each arc of Γ but does not coincide there with any function of E_1, then $f^*(z)$ and $\varphi^*(z)$ are analytic in $\bar{G}$ and $f^*(z)$, if not constant, maps G on an m-sheeted unit circle, $m\geqq n$. Let Γ again consist of analytic curves, and consider the linear functionals on B given by $l_k(f)=\int_\Gamma f(x)\omega_k(x)dx$, $k=1,\cdots,r$, where the $\omega_k(x)$ are analytic on each curve in Γ but no linear combination of them coincides there with a function of E_1. Consider the set A_r of points $C=(C_1,\cdots,C_r)$ in r-space, $C_k=l_k(f)$, $f\in B'$. Then A_r is a closed convex set with the origin as an interior point, and its boundary points correspond to functions mapping G on an m-sheeted circle, $m\geqq n$. If C_k are arbitrary numbers, there is a solution in B of the interpolation problem $l_k(f)=C_k$, $k=1,\cdots,r$; the unique solution whose maximum modulus is smallest maps G on an m-sheeted circle. If l is an additional functional and $(C_1,\cdots,C_r)$ is in A_r, then $\sup|l(f)|$, $f\in B'$, with $l_k(f)=C_k$, is attained for a function mapping G on an m-sheeted circle. There is a similar result for the extremal function for an infinite sum $\sum\rho_k|l_k(f)|^p$ under additional hypotheses. *R. P. Boas, Jr.* (Evanston, Ill.).

Ozawa, Mitsuru. On functions of bounded Dirichlet integral. Kōdai Math. Sem. Rep. **1952**, 95–98 (1952).

The author finds necessary and sufficient (perfect) conditions in terms of local coefficients for a single-valued regular function to have an image domain whose area does not exceed π. Let B be an n-tuply connected domain containing $z=0$ whose boundary consists of analytic curves Γ_ν, $\nu=1,2,\cdots,n$. Let $f_m(z,\alpha)$ be the unique function satisfying the conditions: (a) $f_m(z,\alpha)-1/z^m$ is single-valued and regular in B; and (b) all images of Γ_ν, $\nu=1,\cdots,n$, by $f_m(z,\alpha)$ are line segments with inclination α to the real axis. Suppose that $\frac{1}{2}[f_m(z,0)-f_m(z,\frac{1}{2}\pi)]=\sum_{\nu=1}^\infty S_{m\nu}z^\nu$. The coefficients S_{mn} are called the generalized (m,n)-span. Then a perfect condition that the single-valued regular function $f(z)=\sum_{\mu=1}^\infty c_\mu z^\mu$ satisfy $\int\int_B|f'(z)|^2dxdy\leqq\pi$ is that

$$\left|\sum_{\mu=1}^\infty\mu c_\mu x_\mu\right|^2\leqq\sum_{\mu,\nu=1}^N\nu S_{\mu\nu}x_\nu\bar{x}_\mu\quad(N=1,2,\cdots)$$

hold for arbitrary complex numbers x_μ. *G. Springer.*

Vekua, N. P. The Carleman boundary problem for several unknown functions. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 9–14 (1952). (Russian)

Let L be a simple, closed, smooth contour limiting a bounded domain D^+ in the complex plane ω of $z=x+iy$; $D^-=\omega-(D^++L)$; the origin of the coordinates is in D^+; the angle made by the tangent to L is of class H (Hölder); $\alpha(t)$ is assigned in H on L, $\alpha^{(1)}(t)\neq 0$ and $\alpha(t)$ transforms L one-to-one on itself, changing direction. It is said that $\phi(z)$ is meromorphic in D^+ (in D^-) if ϕ is analytic in D^+ (in D^-), except possibly at a finite number at most of poles, and if ϕ is continuously extendable on L. The author solves the following problem: to find a vector $\phi=(\phi_1,\cdots,\phi_n)$ meromorphic in D^+ so that (1) $\phi^+[\alpha(t_0)]=G(t_0)\phi^+(t_0)+g(t_0)$ (on L), the matrix $G(t_0)=(G_{kj}(t_0))$ $(k,j=1,\cdots,n)$ and the vector $g(t_0)=(g_1,\cdots,g_n)$ being assigned in H. The case $n=1$, $\alpha[\alpha(t)]\equiv t$ has been solved by D. A. Kveselava [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **16**, 39–80 (1948); these Rev. **14**, 152]. The homogeneous problem (1), when $n=1$ and $\alpha[\alpha(t)]\equiv t$, had been considered by T. Carleman [Verh. Internat. Math.-Kongresses, Zürich, 1932, v. 1, Füssli, Zürich-Leipzig, 1932, pp. 138–151]. The author solves the problem completely with the aid of the theory of integral equations in the sense of principal values.

W. J. Trjitzinsky (Urbana, Ill.).

Newns, W. F. On the representation of analytic functions by infinite series. Philos. Trans. Roy. Soc. London. Ser. A. **245**, 429–468 (1953).

This comprehensive paper falls readily into two parts. The first deals with the generalization to a general class of linear spaces of the notion of basic series, originally defined by J. M. Whittaker for special spaces of analytic functions [Sur les séries de base de polynomes quelconques, Gauthier-Villars, Paris, 1949; these Rev. **11**, 344]. In the second part, this specialization is made and new and old results in the classical case follow. The linear spaces studied are special cases of the GF spaces introduced by Donoghue and Smith [Trans. Amer. Math. Soc. **73**, 321–344 (1952); these Rev. **14**, 182] and generalize the $\mathfrak{LF}$ spaces of Schwarz and Dieudonné. For ease of description, the notation of the author has been slightly altered. Let $F_1\subset F_2\cdots$ be a sequence of subspaces of the vector space F such that $\bigcup_1^\infty F_n=F$. Let τ_n and τ be locally convex topologies on F_n and F with $\tau_{n+1}\subset\tau_n$ on F_n, and such that $\langle F_n,\tau_n\rangle$ is an $\mathfrak{F}$ space. If $\{x_k\}\to_\tau x$ is equivalent to the requirement that for some n, $\{x_k\}\subset F_n$ and $\{x_k\}\to_{\tau_n}x$, then $\langle F,\tau\rangle$ is termed an $\mathfrak{F}_+$ space. A base for F is a sequence $\{z_k\}$ such that every $x\in F$ has a unique representation $x=\sum_1^\infty c_k z_k$. When $x\in F_n$, there is an N depending on n but not x such that this series converges in τ_N. A base is then called absolute if $\sum|c_k|\cdot\|z_k\|_p<\infty$ for each pseudonorm $\|\ \|_p$ defining τ_N. Necessary and sufficient conditions that $\{z_k\}$ be a base for F are that for each n, and some N, the linear span of the z_k be τ_N dense in F_n, that there exist continuous functionals $\{Z_n\}$ orthogonal to the $\{z_k\}$, and that the transformations S_n defined by $S_n(x)=\sum_1^n Z_k(x)z_k$ be uniformly bounded. Let $\{p_k\}$ be another sequence of points of F, none 0, and let (π_{kn}) be a matrix such that $z_n=\sum\pi_{kn}p_k$. The pair $\langle\{p_k\},(\pi_{kn})\rangle$ may be called a basic set. Define functionals Π_k by (*) $\Pi_k(x)=\sum\pi_{kn}Z_n(x)$. Then $\Pi_k(z_n)=\pi_{kn}$, so that Π_k is at least defined on the linear space of $\{z_n\}$, which is dense in F. The series (**) $\sum\Pi_k(x)p_k$ is then called the basic series for x. If (*) and (**) converge for all x, then (**) converges to x, and $\{p_k\}$ is said to be effective for F. Assuming that (*) is defined for all x, necessary and sufficient conditions are given in order that (**) converge for all x, in terms of the uniform boundedness of the transformations $\sum_m^n\Pi_k(x)p_k$, which are simplest in case $\{z_n\}$ is an absolute base. A basic set $\{p_k\}$ is called U-basic if the functionals Π_n and p_k are orthogonal. Any U-basic set which is effective for F is a base; conversely, if F has two bases $\{z_n\}$ and $\{p_k\}$, then $\{p_k\}$ is U-basic with respect to $\{z_n\}$. The author applies these results to the space $H(R)$ of functions analytic in $|z|<R$ under the topology of compact convergence, and to $\bar{H}(R)$ the space of functions analytic in $|z|\leqq R$ under the topology α of uniform convergence on $|z|\leqq R$ and β, of uniform convergence on compact sets whose interiors contain $|z|\leqq R$. This portion of the paper contains many known results, as well as new ones, obtained by specialization of the abstract treatment. For example, let $W_n(r)=\max_{\mu},\max_{|z|=r}|\sum_\mu^\nu\pi_{nk}p_k(z)|$, and $K(r)=\limsup\,(W_n(r))^{1/n}$. Then, necessary and sufficient conditions for $\{p_k\}$ to be effective in $H(R)$, $\langle\bar{H}(R),\alpha\rangle$, $\langle\bar{H}(R),\beta\rangle$ are that $K(\rho)<R$ for all $\rho<R$, $K(R)=R$, $\lim_{\rho\downarrow R}K(\rho)=R$. Use is also made of the formal expansion $(w-z)^{-1}=\sum p_k(z)\pi_k(w)$ where $\pi_k(w)=\sum\pi_{nk}w^{-n-1}$ to obtain alternative formulations of these results. Since the basic sets associated with $\{p_k\}$ may be described by means of an infinite matrix and its left inverses, algebraic analogies suggest the consideration of certain basic sets associated with a fixed $\{p_k\}$. These have been considered by Whittaker and others [op. cit.] and their results are obtained in strengthened forms. An error in a paper of Nassif [Amer. J. Math. **69**, 583–591 (1947); these Rev. **9**, 22] is pointed out and corrected. Methods for extending the results to the spaces of functions analytic in certain general domains D are indicated. The case of D simply connected may be handled by using as a base for $H(D)$ the powers of the mapping function for D, or the Faber polynomials when they are defined. The case of D doubly connected is discussed briefly.

R. C. Buck (Madison, Wis.).

Tsuji, Masatsugu. Theory of Fuchsian groups. Jap. J. Math. **21** (1951), 1–27 (1952).

The paper is a study of the equidistribution of equivalent points under a Fuchsian or Fuchsoid group by methods which are for the most part similar to those used in Nevanlinna's theory of meromorphic functions. A sample result of more than routine interest is the following: E is a compact measurable set in the fundamental region, $L(r,E)$ is the non-euclidean linear set of its equivalent points on $|z|=r$, $\Lambda(r,E)$ is an integrated mean of $L(r,E)$, $T(r)$ is an integrated mean of the average number of equivalent points in $|z|<r$, $\sigma(E)$ is the non-euclidean measure of E; then $\lim_{r\to1}\Lambda(r,E)/T(r)=\sigma(E)$, provided that $T(r)\to\infty$.

L. V. Ahlfors (Cambridge, Mass.).

Jacobsthal, Ernst. Über die Klasseninvariante ähnlicher linearer Abbildungen. I. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 119–124 (1953).

With all linear fractional transformations similar to a given one S, in the sense of matrix theory, is associated an invariant Q whose relation to the invariant Q_r associated with S^r is determined.

P. R. Garabedian.

Waadeland, Haakon. Über die Klassen ähnlicher linearer Abbildungen. I. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 125–128 (1953).

Using the results of Jacobsthal [cf. the preceding review], the author determines the classes K^r of transformations

similar to S^r and such that, for all r, only a finite number of the K^r are distinct. *P. R. Garabedian* (Stanford, Calif.).

Waadeland, Haakon. Über die Klassen ähnlicher linearer Abbildungen. II. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 129–130 (1953).

Alternate derivation of the results summarized in the preceding review. *P. R. Garabedian* (Stanford, Calif.).

Komatu, Yûsaku. Einige kanonische konforme Abbildungen vielfach zusammenhängender Gebiete. Proc. Japan Acad. **29**, 1–5 (1953).

Let D be a domain in the complex z-plane which contains $z=\infty$ and is bounded by a finite number of Jordan curves C, and let $C_1 \subset C$, $C_2 = C - C_1$. The author proves the existence of schlicht mappings $f(z)=z+O(1)$ (near $z=\infty$) for which (a) Re $\{f(z)\}$ = const. if $z \varepsilon C_1$, Im $\{f(z)\}$ = const. if $z \varepsilon C_2$; (b) $|f(z)|$ = const. if $z \varepsilon C_1$, arg $\{f(z)\}$ = const. if $z \varepsilon C_2$; (c) as in (b), except that one designated boundary component is carried into a circle. All these mappings are unique and they can be characterized by suitable extremal problems.

Z. Nehari (St. Louis, Mo.).

Komatu, Yûsaku. Mixed boundary value problems. J. Fac. Sci. Univ. Tokyo. Sect. I. **6**, 345–391 (1953).

The mixed boundary value problem treated in this paper is the following: If D is a multiply-connected domain bounded by a finite number of smooth curves $C=C_1+C_2$, to construct a harmonic function from its boundary values on C_1 and its normal derivatives on C_2. The author shows that the role of the Green's and Neumann's functions in the boundary value problem of the first and second kind is here played by a singularity function which is closely related to the mapping function (c) of the preceding review. In the case of the unit circle, this function can be explicitly determined and this leads to a representation of a harmonic function in terms of mixed boundary data similar to Poisson's integral. It is also shown that, for angular approach, this representation shows the same boundary behavior as Poisson's integral. The explicit treatment of the case of the unit circle requires rather formidable computations, which are carried out with patience and fortitude. *Z. Nehari.*

Komatu, Yûsaku, and Mori, Akira. Conformal rigidity of Riemann surfaces. J. Math. Soc. Japan **4**, 302–309 (1952).

T. Radó [Acta Litt. Sci. Szeged **2**, 47–60 (1924)] a démontré ce théorème: Soit G un domaine plan bordé par n ($\geqq 2$) courbes de Jordan C_i et G' un domaine partiel de G, différent de G, et bordé par n courbes de Jordan C_i' telles que C_i' soit homotope à C_i dans $\bar{G}$. Alors G n'admet pas de représentation conforme biunivoque sur G'. L'auteur généralise ce résultat de la façon suivante: Soit G un domaine d'une surface de Riemann, une partie C de la frontière de G étant formée de n courbes de Jordan. Soit G' un domaine partiel de G, H la réunion des composantes connexes de l'ensemble $(G\cup C)-G'$ qui contiennent des points de C, C' la réunion des composantes connexes de la frontière de G' qui appartiennent à H. Supposons que G admette une représentation conforme biunivoque sur G' telle que C corresponde à C'. Alors, ou bien G est conformément équivalent à un disque ou à une disque pointé, ou bien G est de genre infini et il existe une suite non décroissante de domaines G_k de réunion égale à G telle que $C\subset\bar{G}_0$ et que G_k-G_{k-1} admette une représentation conforme biunivoque sur $G_{k+1}-G_k$. Le domaine G admet alors une "composante frontière idéale" unique définie par la suite $G-G_k$; la mesure harmonique de cette composante est nulle, et sa "dimension harmonique" (au sens de M. Heins [Ann. of Math. (2) **55**, 296–317 (1952); ces Rev. **13**, 643]) est l'unité. La démonstration fait intervenir un lemme sur les suites de représentations conformes biunivoques d'une surface de Riemann dans une autre, et sur la métrique hyperbolique de Poincaré. L'auteur donne enfin le résultat suivant: une surface de Riemann de genre positif fini n'admet aucune représentation conforme biunivoque sur une de ses parties distincte d'elle-même. *R. de Possel* (Alger).

Tsuji, Masatsugu. On Ahlfors' theorems on covering surfaces. J. Fac. Sci. Univ. Tokyo. Sect. I. **6**, 319–328 (1953).

Contains modifications of the original proof [Acta Math. **65**, 157–194 (1935)]. *L. V. Ahlfors.*

Sario, Leo. Modular criteria on Riemann surfaces. Duke Math. J. **20**, 279–286 (1953).

If a Riemann surface R is exhausted by compact regions R_n, each $R_{n+1}-R_n$ has a module σ_n. It is well known that R is parabolic if $\prod\sigma_n=\infty$ (the author used exponential modules, thus replacing the more customary sum by a product). Conversely, every parabolic surface has an exhaustion with divergent product. The author shows that there is always exhaustion with a convergent product. Finally, he gives a simpler proof of his criterion for surfaces without Dirichlet bounded analytic functions [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **50** (1948); these Rev. **10**, 365]. *L. V. Ahlfors* (Cambridge, Mass.).

Sario, Leo. Minimizing operators on subregions. Proc. Amer. Math. Soc. **4**, 350–355 (1953).

Let G be a subregion of a Riemann surface bounded by a finite set α of closed analytic Jordan curves. The author constructs several linear operators which solve the Dirichlet problem for G with given values on α and which minimize various functionals related to the Dirichlet integral. This generalizes a previous paper by the author [Trans. Amer. Math. Soc. **72**, 281–295 (1952); these Rev. **13**, 735].

H. L. Royden (Stanford, Calif.).

Habsch, Hans. Die Theorie der Grundkurven und das Äquivalenzproblem bei der Darstellung Riemannscher Flächen. Mitt. Math. Sem. Univ. Giessen no. **42**, i+51 pp. (13 plates) (1952).

This paper deals with simply connected covering surfaces F of the complex z-plane with branch points over a finite number of base points z_i. The points of F over a Jordan curve Γ_1 through the z_i divide F into half-sheets, and F can be represented by a line complex K (Streckenkomplex). The problem under consideration is to determine the change of K if the base curve Γ_1 is replaced by another such curve Γ_2.

Let ϑ be the group of topological self-mappings of the z-plane which permute the z_i. The mappings that are created by a deformation leaving the z_i fixed at each stage form a normal subgroup γ of ϑ. The factor group $L=\vartheta/\gamma$ is determined by the changes $\Gamma_1\to\Gamma_2$ of base curves. Those changes which leave the z_i fixed form a normal subgroup N of L. The simplest changes $\Gamma_1\to\Gamma_2$ are the interchange (Tausch), in which two neighboring z_i on Γ_1 are traversed in the reverse succession by Γ_2, and the intersection (Schnitt), in which two non-neighboring arcs z_iz_j on Γ_1 are intersected once by Γ_2 [Drape, Deutsche Math. **1**, 805–824 (1936)].

The author establishes that the interchanges generate L, the intersections, N. By determining explicitly the effect upon a complex K of an interchange and an intersection, the effect upon K by any change of the base curve Γ can be found. Two systems Γ_1, K_1 and Γ_2, K_2 represent the same Riemann surface if and only if the change $\Gamma_1 \to \Gamma_2$ takes K_1 to K_2.

Line complexes are equivalent if they correspond to the same Riemann surface. Riemann surfaces are termed equivalent if, with base curves suitably chosen, they can be represented by the same line complex. The latter equivalence problem is reduced to the former, and is studied in detail for surfaces with logarithmic branch points. *L. Sario.*

Ahlfors, Lars V. On the characterization of hyperbolic Riemann surfaces. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **125**, 5 pp. (1952).

Ce mémoire ne contient pas de résultat nouveau, mais est un exposé rapide définissant la "surface de Riemann hyperbolique" par l'existence d'une fonction sous-harmonique négative non constante, et démontrant l'équivalence de cette existence avec: (a) la non validité du "principe de maximum", (b) l'existence de la mesure harmonique, (c) l'existence de la fonction de Green. *R. de Possel.*

Gerstenhaber, Murray. On a theorem of Haupt and Wirtinger concerning the periods of a differential of the first kind, and a related topological theorem. Proc. Amer. Math. Soc. **4**, 476–481 (1953).

The author reproves a theorem of Wirtinger which states that a (closed) Riemann surface S_g having an Abelian differential w with just two non-zero periods is necessarily of genus 1. The author uses w to construct an analytic mapping f of S_g onto a torus S_1 such that the images of cycles on S_g over which w has no periods are homotopic to zero on S_1. He then gives two proofs that f is a homeomorphism. One is function-theoretic, the other topological.
H. L. Royden (Stanford, Calif.).

Stoll, Wilhelm. Ganze Funktionen endlicher Ordnung mit gegebenen Nullstellenflächen. Math. Z. **57**, 211–237 (1953).

H. Kneser [Jber. Deutsch. Math. Verein. **48**, Abt. 1, 1–28 (1938)] a montré que le logarithme d'une fonction méromorphe $f(\mathfrak{z})$ $(\mathfrak{z}=(z_1, \cdots, z_n))$ d'ordre fini est représentable, dans un voisinage de l'origine de R^{2n}, au moyen d'une intégrale sur le support du diviseur défini par f. Le but de cet article est de construire, au moyen d'une telle intégrale, une fonction entière de diviseur (positif) donné. De façon précise: L'hypersurface analytique fermée N dans R^{2n} a la multiplicité $\nu(\mathfrak{z})$. Pour un $r_0>0$, supposons que la boule de centre 0 et de rayon r_0 ne rencontre pas N. Alors: 1) Il y a une fonction $q(r)$ possédant les propriétés: a) $q(r)$ est $\geqq 0$, non décroissante et à valeurs entières; b) pour chaque r fixé, l'intégrale

$$\frac{1}{W_{2n-2}}\int_N \nu(\mathfrak{z})\left(\frac{r}{|\mathfrak{z}|}\right)^{q(|\mathfrak{z}|)+1} \partial\omega_{2n-2}(\mathfrak{z}) < \infty$$

où W_{2n-2} est une constante et où $\partial\omega_{2n-2}(\mathfrak{z})$ désigne l'élément d'aire projectif de dimension $2n-2$. 2) Si $q(r)$ est une fonction possédant les propriétés énoncées dans 1), il existe une fonction entière $h(\mathfrak{y})$ qui s'annule seulement sur N avec la multiplicité $\nu(\mathfrak{z})$ et pour laquelle, dans la boule $[|\mathfrak{y}| \leqq r_0]$, on a

$$\text{(1)}\quad \log h(\mathfrak{y}) = \frac{1}{W_{2n-2}}\int_N \nu(\mathfrak{z})\, e\left(\frac{(\mathfrak{y}|\mathfrak{z})}{(\mathfrak{z}|\mathfrak{z})}, q(|\mathfrak{z}|)\right)\partial\omega_{2n-2}(\mathfrak{z});$$

$(a|b)$ désigne le produit scalaire de deux vecteurs a, b de R^{2n} et

$$e(x, q) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\{x^{n-1}\log E(x, q)\},$$

$$E(x, q) = (1-x)\exp\left(\sum_{\lambda=1}^{q}\frac{x^\lambda}{\lambda}\right).$$

Si q est un entier positif tel que l'intégrale de $\nu(\mathfrak{z})|\mathfrak{z}|^{-q-1}\partial\omega_{2n-2}$ sur N à l'extérieur de la sphère de centre 0 et de rayon $\geqq 0$ ait un sens, il existe une fonction $h(\mathfrak{y})$ satisfaisant à (1) avec $q(|\mathfrak{z}|)=q$ (fonction canonique). h étant canonique, on donne une évaluation de $\log|h|$ et des résultats précis sur le type, l'ordre et la classe de h. On appelle surface méromorphe l'image méromorphe d'une variété complexe dans un espace projectif complexe et on démontre un théorème sur le genre et l'ordre d'une telle surface qui s'énonce de la même façon que dans le cas d'une seule variable. La technique repose sur l'intégration sur les variétés complexes étudiées par l'auteur [Math. Z. **57**, 116–154 (1952); ces Rev. **14**, 550]. *P. Dolbeault* (Paris).

Tornehave, H. On analytic functions of several variables. Analytic continuation by Schwarz's reflexion method. Mat. Tidsskr. B. **1952**, 29–37 (1952).

By an example, the author indicates where difficulties arise in trying to carry over Schwarz's reflection method to several complex variables. In spite of this, by jointly using an auxiliary theorem which he derives and the classical theorem on analytic completion for several complex variables, the author is able to obtain a generalization of the reflection method. His auxiliary theorem is the following: Let M_1 and M_2 denote the two components of the open set M defined by the inequalities $x^2=x_1^2+\cdots+x_m^2<ma^2$, $0<y^2=y_1^2+\cdots+y_m^2<ma^2$, $1/k<y_j/y_1<k$, $j=2, \cdots, m$, where $a>1$, $k>1$. Let S denote the sphere $x^2<ma^2$, $y=0$. Let $f(z)$ be continuous in $M \cup S$ and analytic in each of the domains M_1 and M_2. Then there exist a spherical neighborhood U of the point $z=0$ and an analytic continuation of f into U. Having this theorem, he then obtains his two main theorems. One of these is a general theorem on analytic continuation of a function which is analytic in two domains and continuous in an m-dimensional set of points that belong to the real subspace and to the boundary of both domains. The other theorem is his main result on analytic continuation by Schwarz's reflection method. *W. T. Martin.*

Tornehave, H. On analytic functions of several variables. Some results concerning analytic completion. Mat. Tidsskr. B. **1952**, 44–60 (1952).

In the space of m complex variables $z_j=x_j+iy_j$, $j=1, \cdots, m$, consider a cylindrical domain $(\omega_1, \cdots, \omega_m)$ where each two-dimensional domain ω_j is simply connected and symmetric with respect to the real axis, and consider what the author calls a "skeleton" tube $[R_m; C]$ where R_m is the m-dimensional real space of $x_1, \cdots, x_m$ and C is the axial cross $\sum_{1\leqq\mu<\nu\leqq m} y_\mu^2 y_\nu^2$ formed by the coordinate axes in the y-space. Denote by Ω the intersection of these two sets. The author obtains the analytic completion of such sets Ω. He first considers the case in which each ω_j is a half-plane $x_j>0$. In this case, he proves that Ω has an analytic completion defined by the inequalities $\sum_1^m |\arg z_j| < \pi/2$. He bases the proof of his principal theorem on this special result. For the general case, with each domain ω_j he associates a function $\phi_j(z_j)$. If ω_j is the whole plane, ϕ_j is defined to be identically one. If ω_j is not the whole plane, denote

by L_j the segment of the real axis contained in ω_j and denote by $\zeta_j=\phi_j(z_j)$ a function which maps ω_j on the half-plane $\operatorname{Re}\zeta_j>0$ such that L_j corresponds to the positive real axis. With this notation, the main theorem states that the analytic completion of such a set Ω is the domain defined by the conditions $z_j \in \omega_j$, $j=1, \cdots, m$, $\sum_1^m|\arg \phi_j(z_j)|<\pi/2$. The proof is based on an idea contained in S. Bochner's proof of his theorem on analytic completion of tubes [Ann. of Math. (2) 39, 14–19 (1938)]. Having this general theorem, the author also obtains another theorem on analytic completion which he combines with one of the theorems contained in the paper reviewed in the preceding review. The combined result yields a more general result on analytic completion by the reflection method. *W. T. Martin.*

Thimm, Walter. Über die Menge der singulären Bildpunkte einer meromorphen Abbildung. Math. Z. 57, 456–480 (1953).

Un point $x^0=(x_i^0)$ est dit point d'indétermination pour une représentation méromorphe $z=A_k(x)$ de l'espace $C^n(x_i)$ dans l'espace $C^k(z_j)$, définie par

$$(1)\qquad z_j=\frac{p_j(x_i)}{q_j(x_i)},\quad i=1, \cdots, n;\ j=1, \cdots, k,$$

p_j, q_j fonctions holomorphes en x^0, si, pour un indice j au moins, on a $p_j(x^0)=q_j(x^0)=0$. Un point ζ de C^k est dit image singulière de x^0 par (1) s'il est valeur limite de la transformation, c'est-à-dire s'il existe une suite $x^{(\nu)}\rightarrow x^0$, $z^{(\nu)}=A_k[x^{(\nu)}]\rightarrow\zeta$, les $x^{(\nu)}$ n'étant pas des points d'indétermination de $z=A_k(x)$. Le mémoire étudie l'ensemble $K(x^0)$ des images singulières d'un point d'indétermination unique x^0 et apporte un premier résultat en donnant une démonstration d'un énoncé aperçu par Autonne [Acta Math. 21, 249–263 (1897)]: $K(x^0)$ est constitué par une variété algébrique. L'étude faite va plus loin: dans le cas où $z=A_k(x)$ est appliquée à un voisinage de x^0 dans C^n, tout point de $K(x^0)$ est de première espèce, c'est-à-dire est limite de la transformation sur une courbe γ: $x_i=x_i(t)$, $x_i(t)$ holomorphe $t=0$, $x_i(0)=x_i^0$, sur laquelle aucun des $p_j[x_i(t)]$, $q_j[x_i(t)]$ ne s'annule identiquement. Plus généralement le mémoire étudie l'image $A_k(\Delta)$ dans C^k d'une variété Δ de C^n, définie voisinage x^0 par $\phi_\nu(x)=0$, $\nu=1, \cdots, N$, les ϕ_ν étant holomorphes et nulles en x^0: $K(x^0, \Delta)$ est encore une variété algébrique; elle peut être construite à partir d'un système

$$(2)\qquad x_\lambda=\sum_{p=1}^{p=r} a_{\lambda,p}t^p$$

de polynomes en t dont les coefficients satisfont des équations algébriques

$$(3)\qquad A_\mu(a_{1,1}, \cdots, a_{n,r})=0,\quad \mu=1, \cdots, N';$$

en substituant (2) dans les équations (1) et passant à la limite $t\rightarrow 0$ (le cas où certaines des p_j, q_j s'annulent identiquement sur la courbe (2) exige quelques précautions); ce procédé donne de la variété algébrique (3) une représentation rationnelle dans $K(x^0, \Delta)$; le résultat est étendu à l'espace projectif. *P. Lelong* (Lille).

Bremermann, Hans-Joachim. Die Charakterisierung von Regularitätsgebieten durch pseudokonvexe Funktionen. Schr. Math. Inst. Univ. Münster no. 5, i+92 pp. (1951).

In the space of n complex variables $z_1, \cdots, z_n$ $(n>1)$ not every domain is a domain of regularity, that is, the existence domain of a regular function, but every domain G possesses a smallest domain containing it which is a domain of regularity. Such a domain is called the regularity envelope of G.

The main tool of the author in the present investigation of domains of regularity is the concept of what he and K. Oka call pseudo-convex functions (these functions have been called "plurisubharmonic" by P. Lelong, bisubharmonic by G. O. Thorin, and Hartogs' functions by S. Bochner and the reviewer). For $n=1$ these functions are the classical subharmonic functions. For $n>1$ a real-valued function $V(z_1, \cdots, z_n)$ is called pseudo-convex in a domain G of R_{2n} if it is upper semi-continuous, and is subharmonic on every two-dimensional analytic plane which cuts G.

In Chapter I the author first carries over to subharmonic functions on Riemann surfaces many of the classical results of subharmonic functions $(n=1)$. He then summarizes those parts of the works of the above named authors on pseudo-convex functions which he needs and he develops the theory further for use in later chapters.

In Chapter II he investigates the connection of pseudo-convex functions with domains of regularity. Consider a Hartogs series $\sum_1^\infty f_\nu(z_1, \cdots, z_n)w^\nu$ where the f_ν are all single-valued and regular in a domain G of R_{2n}. Denote by $R(z)=e^{-V(z)}$ the regularity radius of the series and denote by H the domain (in R_{2n+2})$[|w|<R(z),\ z\in G]$. As a key result the author proves that H is a domain of regularity in the space of $z_1, \cdots, z_n, w$ if and only if (1) $V(z)$ is pseudo-convex and (2) G is a domain of regularity in z-space (special cases and less general versions of this result have been obtained earlier by various authors including the important result obtained by Hartogs [Math. Ann. 62, 1–88 (1906)]).

In the third chapter the author uses the continuity theorem in the construction of regularity envelopes of certain types of domains, in particular, in the construction of regularity envelopes of semi-tubes. *W. T. Martin.*

Lelong, Pierre. Domaines convexes par rapport aux fonctions plurisousharmoniques. J. Analyse Math. 2, 178–208 (1952).

Ein schlichtes Gebiet D im Raume C^n von n komplexen Veränderlichen heisst nach Lelong P-konvex, wenn es zu jedem ganz in D liegenden Gebiet Δ, $\Delta\subset D$, ein Gebiet $\Delta'\supseteq\Delta$ gibt, sodass a) $\Delta'\subset D$ und b) es zu jeder offenen Menge ω aus $D-\overline{\Delta'}$ einen Punkt P und dazu eine in D plurisubharmonische Funktion V gibt [zur Definition der plurisubharmonischen Funktion siehe P. Lelong, J. Math. Pures Appl. (9) 31, 191–219 (1952); diese Rev. 14, 463], sodass $V(P)>V(\Delta)$. Die P-Konvexität von D ist invariant gegenüber ein-eindeutigen analytischen Abbildungen. Sind D_1 und D_2 P-konvex, so ist es auch jede zusammenhängende Komponente von $D_1\cap D_2$. D heisst lokal P-konvex in einem Punkte M, wenn es eine $2n$-dimensionale Hyperkugel H um M gibt, sodass $D\cap H$ aus P-konvexen Gebieten besteht. Die lokale P-Konvexität ist offenbar nur in den Randpunkten M von D fraglich. D heisst schlechtweg lokal konvex, wenn es in jedem Punkte M von C^n lokal konvex ist.

Hauptsatz der vorliegenden Arbeit: Wenn D lokal P-konvex ist, so ist D auch P-konvex. Dazu werden eine Reihe von Korollaren gegeben, so: 1) Damit D P-konvex ist, ist notwendig und hinreichend, dass der Schnitt von D mit jedem linearen Unterraum von 2 komplexen Veränderlichen nur P-konvexe Komponenten hat. 2) Die P-Konvexität von D ist äquivalent mit der Existenz einer in D plurisubharmonischen Funktion $V(P)$, die gegen $+\infty$ geht, wenn P sich dem Rande von D nähert. 3) D ist P-konvex, wenn der spezielle Kontinuitätssatz (für analytische Ebenen) in bezug auf den Rand von D gültig ist. 4) D ist P-konvex, wenn D pseudokonvex von aussen im ursprünglichen Sinne von

E. E. Levi ist. [Siehe dazu die Arbeiten, die den Anfang für alle diese Untersuchungen bilden: E. E. Levi, Ann. Mat. Pura Appl. (3) **17**, 61–87 (1910); **18**, 69–79 (1911).]

H. Behnke (Münster).

Bochner, S., and Martin, W. T. Hartogs' theorem in complex spaces with singularities. J. Indian Math. Soc. (N.S.) **16**, 137–146 (1952).

Théorème A: Soient U et V deux variétés analytiques complexes, et $f(x, y)$ une fonction sur $U\times V$ telle que, pour chaque $a \in U$, $f(a, y)$ soit holomorphe en $y \in V$, et, pour chaque $b \in V$, $f(x, b)$ soit holomorphe en $x \in U$. Alors $f(x, y)$ est holomorphe dans $U\times V$. Théorème B: Soient D et $\bar{D}$ deux ouverts de $U\times V$, et, pour chaque point $a \in U$, soient D_a et $\bar{D}_a$ les sections de D et $\bar{D}$. Supposons que, pour tout a, chaque composante connexe de $\bar{D}_a$ contienne au moins une composante connexe de D_a. Si $f(x, y)$ est holomorphe dans D et si, pour chaque $a \in U$, $f(a, y)$ est holomorphe dans $\bar{D}_a$, alors $f(x, y)$ est holomorphe dans $\bar{D}$.

Ces deux théorèmes sont dus essentiellement à Hartogs [Math. Ann. **62**, 1–88 (1906)]. Les auteurs les étendent au cas où U et V sont des variétés analytiques complexes dans un sens généralisé, défini dans l'article analysé ci-dessous. Il faut alors distinguer entre les deux notions de "fonction holomorphe" et "fonction strictement holomorphe".

H. Cartan (Paris).

Bochner, S., and Martin, W. T. Complex spaces with singularities. Ann. of Math. (2) **57**, 490–516 (1953).

La notion de variété analytique complexe V (de dimension complexe n quelconque) a besoin d'être élargie, par l'introduction de singularités algébroïdes considérées en quelque sorte d'une manière interne à V. Pour $n=1$, on connaît bien les points de ramification algébrique (isolés) d'une surface de Riemann; mais, du point de vue abstrait, leur introduction est inutile, car le voisinage d'un tel point est "isomorphe" au voisinage d'un point ordinaire. Il n'en est plus de même pour $n\geqq 2$, comme bien connu. Ce mémoire présente un essai d'une théorie générale. Dans la Partie I, les auteurs tentent de définir localement la notion d'"espace complexe avec singularités"; mais leur traitement est, en fait, intermédiaire entre le local et le global. Un rôle essentiel est joué par un certain espace topologique U, dont la définition n'a pas parue claire au rapporteur; il semble que U soit simplement une sous-variété analytique (avec singularités algébroïdes) dans un ouvert d'une espace numérique complexe, mais que les auteurs ne tentent pas de dissocier les différentes composantes irréductibles de U qui passent en un même point; cette sous-variété est d'ailleurs astreinte à des conditions restrictives dont l'énoncé prend une page entière, et qui sont liées à un mode particulier de définition de cette sous-variété. On définit ensuite la notion fondamentale de "fonction holomorphe" sur un tel espace U; le rapporteur ne peut pas comprendre comment les définitions données sont compatibles avec le théorème 3.1, car une fonction satisfaisant aux hypothèses de ce théorème ne prend pas nécessairement une valeur unique en un point de U où passent plusieurs composantes irréductibles. Enfin, les auteurs introduisent la notion de fonction "strictement holomorphe" dans U; il semble qu'une telle fonction soit simplement une fonction induite sur la sous-variété U par une fonction holomorphe de l'espace ambiant.

Dans la Partie III, en recollant ensemble plusieurs morceaux du type précédent, les auteurs tentent de définir la notion globale d' "espace complexe avec singularités". Mais ils ne disent pas quand deux modes de définition définissent des espaces isomorphes. Plus tard, ils utilisent la notion de fonction "strictement holomorphe", qui, selon l'opinion du rapporteur, est dépourvue de sens, car elle ne se conserve pas par recollage des morceaux. Puis les auteurs étendent à ces fonctions strictement holomorphes les résultats antérieurs de Bochner sur la dépendance linéaire et algébrique des fonctions automorphes [J. Indian Math. Soc. (N.S.) **16**, 1–6 (1952); ces Rev. **13**, 932].

La Partie I contient des théorèmes sur la convergence uniforme des familles équicontinues de fonctions strictement holomorphes; et la Partie II est consacrée aux fonctions algébroïdes dans un ouvert S de l'espace numérique C^k. Quand S est la boule-unité, on a une généralisation du lemme de Schwarz. En général, si $\hat{f}(z)$ désigne la plus grande des valeurs absolues des déterminations de la fonction aglébroïde $f(z)$, $\log \hat{f}(z)$ est pluri-sousharmonique au sens de Lelong (qui n'est pas cité); on en tire quelques conséquences.

H. Cartan (Paris).

Bochner, S., and Martin, W. T. Local transformations with fixed points on complex spaces with singularities. Proc. Nat. Acad. Sci. U. S. A. **38**, 726–732 (1952).

Cette note fait suite à le mémoire antérieur analysé ci-dessus. Nous renvoyons au précédent compte-rendu dont nous conservons les notations. Les auteurs proposent une définition pour la notion d'application holomorphe de U dans U; malheureusement, pour prouver que la composée de deux applications holomorphes est holomorphe, il faut des hypothèses restrictives (th. 1). Le th. 2 est un lemme en vue du th. 3, qui concerne un groupe compact Γ de transformations holomorphes de U dans U. De par sa définition, U est munie d'une projection P sur une boule ouverte S d'un espace numérique, de manière que l'image réciproque de tout point de S se compose d'un nombre fixe de points de U; si on suppose l'existence d'un point $a \in U$ tel que, pour toute transformation T du groupe Γ, l'application PT de U dans S soit tangente à P au point a, le th. 3 dit qu'il existe une application holomorphe R de U dans S, tangente à P au point a, et telle que $RT=R$ pour tout $T \in \Gamma$. On le prouve en faisant une moyenne grâce à la mesure de Haar de Γ.

H. Cartan (Paris).

Esteban Carrasco, Luis. The geometry associated with the third derivative of a polygenic function. Collectanea Math. **4**, no. 2, 121–199 (1951). (Spanish)

This is a contribution to the theory of polygenic functions as developed by Kasner [the paper provides a fairly comprehensive bibliography]. In the first part of the paper, the author gives a brief historical introduction to the subject, and discusses the first and second derivatives of a polygenic function. Some of the theorems of Kasner are reviewed, the most fundamental of which is that the first derivative of a polygenic function at a fixed point $z=x+iy$, is represented by a circle, the Kasner circle, or more comprehensively by a Kasner clock. Also, some theorems of Kasner and De Cicco are studied, perhaps the most important of which is that the second derivative of a polygenic function at a point z, is represented by a congruence of limaçons.

In the second part of the paper, the author develops formulas for the third derivative of a polygenic function. In general, the derivative of order n depends on differential elements of order n. A theorem of Kasner is that if the derivative of order n of a polygenic function depends on differential elements of order at most $(n-1)$ in a region of

the complex z-plane, then the function is monogenic in that region. Also for a fixed differential element of order $(n-1)$, this derivative of order n is represented by a straight line. The author establishes the fact that the rectilinear third derivative is represented by an algebraic curve of order six and class six. After giving a rather extensive study of such an algebraic curve, the circular third derivative for a fixed lineal element is shown to be represented by a parabola. At a fixed point z the circular third derivative induces a rational transformation T whose inverse T^{-1} is algebraic and irrational. For the parabolic third derivative, the straight lines mentioned above envelope an algebraic curve of order twelve. The paper concludes with the discussion of some of the theorems mentioned above concerning the derivative of order n. *J. De Cicco* (Chicago, Ill.).

Theory of Series

Meyer, Burnett. On the convergence of alternating double series. Amer. Math. Monthly **60**, 402–404 (1953).

A double series $\sum_{i,j=1}^{\infty} a_{ij}$ is called diagonally summable if $\sum_{k=2}^{\infty} D_k$ converges, where $D_k=\sum_{i+j=k} a_{ij}$; and it is alternating if each row and column is an alternating simple series. Moreover, an alternating series is called monotonic if $|a_{ij}|\leqq|a_{mn}|$ for $i\geqq m$, $j\geqq n$. It is shown that the monotonic alternating series $\sum a_{ij}$ is σ-convergent if and only if it is diagonally summable. *I. M. Sheffer.*

Hitotumatu, Sin. On the convergence of a multiple power series. Kōdai Math. Sem. Rep. **1952**, 111–114 (1952).

The author says that a double series $\sum_{m,n=0}^{\infty} a_{mn}$ is P-convergent with sum s, if the partial sums $s_{mn}=\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} a_{\mu\nu}$ converge to s in the sense that for every $\epsilon>0$ there is a positive number $i_0=i_0(\epsilon)$ such that $|s_{mn}-s|<\epsilon$ for all $m, n>i_0$. He says that the series is A-convergent (A refers to arrangement) if at least one of the simple series into which the original series can be arranged is convergent. He gives simple examples of (i) a series which is A-convergent but not P-convergent, (ii) a series which is P-convergent but not A-convergent, and (iii) a series which is both A- and P-convergent but not absolutely convergent. As his main result, the author proves the following theorem. Let the double power series $\sum_0^{\infty} b_{mn}x^m y^n$ be P-convergent at every point of a neighborhood of a point (x_0, y_0). Then it converges absolutely and uniformly in $|x|<|x_0|$, $|y|<|y_0|$. As the author points out, the double power series for the function $(2-y)/(1-x)+(2-x)/(1-y)$ is P-convergent at $x=2$, $y=2$ yet its absolute convergence region is not $|x|<2$, $|y|<2$ but is $|x|<1$, $|y|<1$. *W. T. Martin.*

Srivastava, R. S. L. On a class of method of summability. Ganita **3**, 71–77 (1952).

According to the author a summability method is of type C if it can be expressed as the sum of two or more methods, one of which is regular and the others are multiplicative with multiplier zero (multiplicatively null). He gives a formal proof of the obvious fact that methods of type C are regular if there are only a finite number of summands; and he states the following generalization, namely, that $(A_{nk})\equiv(a_{nk})+\sum_r(b^r{}_{nk})$ will be regular if (a_{nk}) is regular, and if the $(b^r{}_{nk})$ are multiplicatively null and such that $\sum_r M_r<\infty$, where $M_r=\sup_n \sum_k |b^r{}_{nk}|$. He shows that the Riesz method $(R, \log n, 1)$ is a regular method of type C, and that it is equivalent to the Riesz method $(R, 1)$ for all sequences $\{s_n\}$ such that $s_n=o(\log n)$. He concludes with the following statements, both of which are false as can be seen by choosing (a_{nk}), for example, as equivalent to the identity. (a) If a method of summability (a_{nk}) is translative, then it evaluates the sequence $\{(-1)^k\}$ to zero. (b) If (a_{nk}) is translative, then the method (b_{nk}), with $b_{nk}\equiv a_{nk}+\sigma(-1)^k a_{nk}$ $(\sigma\neq 0)$, will be regular but not translative. Both statements become true if it is assumed, for example, that (a_{nk}) evaluates $\{(-1)^k\}$ to some limit L; and it then follows easily that $L=0$. *J. D. Hill* (East Lansing, Mich.).

Rajagopal, C. T. On the relation of limitation theorems to high-indices theorems. J. London Math. Soc. **28**, 322–329 (1953).

Let $\varphi(u)$ be positive, continuous, and monotone decreasing over $u\geqq 0$, let $\varphi(0)=1$, let $\varphi(\infty)=0$, and let

$$u|\varphi(u')-\varphi(u)|/(u'-u)<M$$

when $0<u<u'$. Let $1<\lambda_1<\lambda_2<\cdots$ and let $\lambda_n\to\infty$. Let $u_1, u_2, \cdots$ be a sequence of positive numbers such that, for each fixed $t>0$, $u_n\varphi(\lambda_n t)\to 0$ as $n\to\infty$. Suppose finally that the condition $a_n=O(u_n)$ is necessary in order that the series $\sum a_n$ may be such that the series $\sum\varphi(\lambda_n t)a_n$ converges when $t>0$ to a function $\Phi(t)$ which has a finite limit as $t\to 0+$. Then

$$\max\left\{\frac{\lambda_n}{\lambda_{n+1}-\lambda_n}, \frac{\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}\right\}=O(u_n).$$

Applications are given for the transformation of Riesz, Laplace-Abel, Stieltjes, and Lambert. *R. P. Agnew.*

de Bruijn, N. G., and Erdös, P. On a recursion formula and on some Tauberian theorems. J. Research Nat. Bur. Standards **50**, 161–164 (1953).

A given sequence $c_1, c_2, \cdots$ for which $c_k\geqq 0$ and $\sum_{k=1}^{\infty} c_k=\infty$ determines a sequence $f(1), f(2), \cdots$ by the formulas $f(1)=1$ and $f(n)=\sum_{k=1}^{n-1} c_k f(n-k)$ when $n>1$. Numerous theorems and conjectures involve hypotheses, some of which are Tauberian, on the sequence c_k and conclusions on the sequence $f(n)$. *R. P. Agnew* (Ithaca, N. Y.).

Fernandes Costa, M. A. Some theorems on limits of sequences. Gaz. Mat., Lisboa **14**, no. 54, 7–13 (1953). (Portuguese)

Expository paper.

Forster, Herbert. Ein Grenzwertsatz. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 93–97 (1953).

Let $F(x)=\sum_{n=0}^{\infty} f_n(x)$ where $f_n(x)=b_0 b_1\cdots b_n x^{b+cn}$, $b_n\downarrow 0$, and $\lim_{n\to\infty} b_{n+k}/b_n=1$ for each k. Let $G(x)=\sum_0^{\infty} g_n(x)$ where $g_n(x)>0$ for large n, and $\lim_{x\to+\infty}|g_n(x)/g_{n+1}(x)|=0$, for each n. If τ_n is a sequence of real numbers, $0\leqq\tau_n\leqq 1$, such that $\lim_{n\to\infty} f_n(x)(b_n x^c)^{\tau_n}/g_n(x)=K$ uniformly for $x\geqq x_0$, then $\lim_{x\to\infty} F(x)/G(x)=K$. As an example, let

$$F(x)=\sum_0^{\infty} x^{2n}/n!\Gamma(n+\alpha+1)\Gamma(2n+\alpha+1)$$

and $G(x)=x^{1/2}F'(x)$. Using the theorem with $\tau_n=3/4$, it follows that $\lim_{x\to+\infty} F(x)/G(x)=2^{-1/2}$. *R. C. Buck.*

Carlitz, L. Remark on a formula for the Bernoulli numbers. Proc. Amer. Math. Soc. **4**, 400–401 (1953).

Using divergent series, Garabedian [Bull. Amer. Math. Soc. **46**, 531–533 (1940); these Rev. **2**, 88] proved the

following formula:

$$(*)\qquad B_{k+1}=\frac{(-1)^{k+1}(k+1)}{2^{k+1}-1}\sum_{r=0}^{k}(-1)^r\frac{\Delta^r 1^k}{2^{r+1}}\quad(k\geqq 0),$$

where the even suffix notation is employed for the Bernoulli numbers. The present author points out that (*) appears in a paper of Worpitzky [J. Reine Angew. Math. 94, 203–232 (1883), formula (68)] and then gives a short proof of (*).
A. L. Whiteman (Princeton, N. J.).

García Pradillo, Julio. Relation between the ordinary and iterated limits of doubly infinite sequences. Gaceta Mat. (1) 5, 8–10 (1953). (Spanish)

Janković, Zlatko. Two recurrence formulae for the sums s_{2k}. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 8, 27–29 (1953). (Serbo-Croatian summary)

The sums in question are $s_{2k}=\sum_{n=1}^{\infty}n^{-2k}$.

Fourier Series and Generalizations, Integral Transforms

Fejér, Leopold. Eigenschaften von einigen elementaren trigonometrischen Polynomen, die mit der Flächenmessung auf der Kugel zusammenhängen. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 62–72 (1952).

This is a discussion, replete with instructive historical references, of the following two trigonometric polynomials

$$\varphi(\theta)=\sin\theta+\sin 2\theta+\cdots+\sin(n-1)\theta+\tfrac{1}{2}\sin n\theta,$$

$$\vartheta(\theta)=\sin\theta+\tfrac{1}{2}\sin 2\theta+\cdots+\frac{1}{n}\sin n\theta.$$

$\varphi(\theta)$ was summed by Archimedes in his famous Proposition 22 [see T. L. Heath, The works of Archimedes, Cambridge, 1897, p. 29] of his treatise on the sphere and cylinder. Fejér observes that $\varphi(\theta)/\sin\theta$ differs by a positive constant from the so-called Fejér kernel, hence $\varphi(\theta)>0$ for $0<\theta<\pi$. The paper concludes with a new and remarkable proof of the positivity of $\vartheta(\theta)$ in the range $0<\theta<\pi$.
I. J. Schoenberg (Philadelphia, Pa.).

Men'šov, D. On limits of indeterminacy of partial sums of trigonometric series. Ann. Soc. Polon. Math. 25 (1952), 323–337 (1953). (Russian)

The author considers the behavior of the nth partial sums $S_n(x)$ of a trigonometric series

$$(*)\qquad \tfrac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$$

with coefficients a_n, b_n tending to zero and proves the following two theorems. (1) There exists a series (*) such that for any increasing sequence of integers n_k we have

$$\liminf S_{n_k}(x)=-\infty,\quad \limsup S_{n_k}(x)=+\infty$$

almost everywhere. (2) Given any two measurable functions $F(x)$ and $G(x)$ of period 2π and satisfying the inequality $G(x)\leqq F(x)$, there exists a series (*) such that $F(x)$ and $G(x)$ are respectively the upper and the lower limits in measure [definition follows] of the partial sums $S_n(x)$ and that, for any increasing $\{n_k\}$,

$$\liminf S_{n_k}(x)\leqq G(x)\leqq F(x)\leqq \limsup S_{n_k}(x)$$

almost everywhere. The author says that a measurable function $F(x)$ is the upper limit in measure of a sequence of measurable functions $g_k(x)$, $a\leqq x\leqq b$, if a) for any measurable function $\varphi(x)$ the set of points where simultaneously $g_k(x)>x$, $\varphi(x)>F(x)$, is of measure tending to zero as $k\to\infty$; b) for any measurable function $\psi(x)$ such that $\psi(x)<F(x)$ in a set of positive measure, the set of points where simultaneously $g_k(x)>\psi(x)$, $F(x)>\psi(x)$ is of measure not tending to zero. The lower limit in measure is defined correspondingly. The author also shows that (3) if $F(x)$ and $G(x)$ are respectively the upper and the lower limits in measure of a sequence of functions $g_k(x)$, then almost everywhere the interval $(G(x), F(x))$ is comprised in the interval $(\liminf g_k(x), \limsup g_k(x))$. [Theorem (1) is not new. Since selecting a subsequence from a given sequence $\{S_n(x)\}$ is an application of a linear method of summation, any lacunary series which is not in L^2 (e.g., the series $\sum n^{-\frac{1}{2}}\cos 2^n x$) satisfies the conclusions of Theorem (1). See the reviewer's paper in Trans. Amer. Math. Soc. 34, 435–446 (1932)].
A. Zygmund (Chicago, Ill.).

Stečkin, S. B. On absolute convergence of Fourier series. Izvestiya Akad. Nauk SSSR. Ser. Mat. 17, 87–98 (1953). (Russian)

A well-known theorem of S. Bernstein [C. R. Acad. Sci. Paris 199, 397–400 (1934)] states that if the periodic function $f(x)$ has modulus of continuity $\omega(\delta, f)$, if (1) $\omega(\delta, f)=O(\omega(\delta))$, and (2) $\sum n^{-\frac{1}{2}}\omega(1/n)$ converges, then the Fourier series of $f(x)$ converges absolutely. The author gives a complete solution of the problem suggested by Bernstein's theorem: he constructs a certain majorant $\omega^*(\delta)$ of $\omega(\delta)$ and shows that (1) implies the absolute convergence of the Fourier series if and only if $\sum n^{-\frac{1}{2}}\omega^*(1/n)$ converges. The more difficult part of the theorem is, of course, the construction of a counterexample when the series diverges. To construct the majorant ω^*, put

$$\omega^{(0)}(0)=0,\quad \omega^{(0)}(\delta)=\inf_{\delta\leqq\eta\leqq\pi}\omega(\eta)\quad(0<\delta\leqq\pi);$$

then

$$\omega^{(1)}(\delta)=\inf_{0\leqq h\leqq\delta}\{\omega^{(0)}(h)+\omega^{(0)}(\delta-h)\}\quad(0\leqq\delta\leqq\pi),$$

and $\omega^{(k)}(\delta)$ $(k>1)$ is obtained in the same way from $\omega^{(k-1)}(\delta)$. Finally, $\omega^{(k)}(\delta)$ decreases to a limit as $k\to\infty$, and this limit is $\omega^*(\delta)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Matsuyama, Noboru. On the convergence of the Fourier series of $|f(x)|$ at one point. Mem. Fac. Sci. Kyūsyū Univ. A. 6, 107–112 (1952).

L'auteur construit une fonction paire $f(x)$ dont la série de Fourier est sommable $(C, 0)$ (ou $(C, 1)$) au point $x=0$, celle de $|f(x)|$ ne l'étant pas. *M. Zamansky* (Paris).

Shapiro, Victor L. Square summation and localization of double trigonometric series. Amer. J. Math. 75, 347–357 (1953).

Let $T=\sum a_M e^{iMX}$ be a double trigonometric series where $M=(m, n)$, $X=(x, y)$, and $MX=mx+ny$. If the sequence

$$\sigma_R{}^\rho(X)=\sum_{|M|\leqq R}(1-|M|^2/R^2)^\rho a_M e^{iMX}\quad(\rho>0),$$

$|M|$ being max $(|m|, |n|)$, converges to a limit s, then T is said to be square summable (C, ρ) to s. Concerning this summation, the author develops the theory of formal multiplication and the principle of localization due to Rajchman and Zygmund [Zygmund, Trigonometrical series, Warszawa-

Lwów, 1935, §11.4–11.46, pp. 278–287]. Among others, he proves the following theorem: If T and T' are two double trigonometrical series with coefficients of order

$$o[(|m|+1)^{-\gamma}(|n|+1)^{-\eta}] \quad (0\leqq\gamma+\eta\leqq1,\ 0\leqq\gamma<1,\ 0\leqq\eta<1)$$

and if the function

$$F(X)=\frac{a_0}{4}x^2y^2+\frac{y^2}{2}\sum_{m\neq0}\frac{a_{m0}}{(im)^2}e^{imx}+\frac{x^2}{2}\sum_{n\neq0}\frac{a_{0n}}{(in)^2}e^{iny}+\sum_{m\neq0,\,n\neq0}\frac{a_M}{(mn)^2}e^{iMX}$$

and $F'(X)$, defined similarly by T', are equal in a closed set R in the interior of the square $0\leqq x<2\pi$, $0\leqq y<2\pi$, then the series $T-T'$ is uniformly square summable $(C, 1-\gamma-\eta)$ to zero in every closed set contained in the interior of R.

S. Izumi (Tokyo).

Žak, I. E. Concerning a theorem of L. Cesari on conjugate functions of two variables. Doklady Akad. Nauk SSSR (N.S.) 87, 877–880 (1952). (Russian)

Let $f(x, y)$ be any real function, periodic of period 2π with respect to x and y, L-integrable in the square $Q=[0, 0; 2\pi, 2\pi]$, and let $\sum A_{mn}(x, y)$ be the double Fourier series of $f(x, y)$. Let $f_1(x, y)$, $f_2(x, y)$, $f_3(x, y)$ be the three conjugate functions of f defined, as in the one-dimensional case, by

$$f_1=(2\pi)^{-1}\int\varphi_1(s; x, y)\operatorname{ctg}(s/2)ds,$$

$$f_2=(2\pi)^{-1}\int\varphi_2(t; x, y)\operatorname{ctg}(t/2)dt,$$

$$f_3=(2\pi)^{-2}\int\int\varphi_3(s, t; x, y)\operatorname{ctg}(s/2)\operatorname{ctg}(t/2)dsdt,$$

where each integral ranges from 0 to π, and

$$\varphi_1=f(x+s, y)-f(x-s, y),$$
$$\varphi_2=f(x, y+t)-f(x, y-t);$$
$$\varphi_3=f(x+s, y+t)-f(x-s, y+t)-f(x+s, y-t)+f(x-s, y-t).$$

I) If f is Lip α in Q or in any closed square interior to Q, then f_1, f_2, f_3 satisfy the following relations

(1) $|f_1(x+h, y+k)-f_1(x, y)|=O(h^\alpha)+O(k^\alpha|\log k|)$;
(2) $|f_2(x+h, y+k)-f_2(x, y)|=O(h^\alpha|\log h|)+O(k^\alpha)$;
(3) $|f_3(x+h, y+k)-f_3(x, y)|=O(h^\alpha|\log h|)+O(k^\alpha|\log k|)$.

As a corollary we have: II) If f is Lip α in Q, or Q', then f_1, f_2, f_3 are Lip α' for all $0<\alpha'<\alpha$ in Q, or Q', respectively. An example is given showing that the functions f_1, f_2, f_3 may be not Lip α. Both statements above extend to double Fourier series a theorem due to Privaloff on simple Fourier series. Statement II has been already proved by the reviewer [Ann. Scuola Normale Super. Pisa (2) 7, 279–295 (1938)]. [The author sets right also an incorrect statement of the reviewer concerning functions f_1, f_2. The correction does not affect the validity of the proof, given for f_3 only, but which holds also for f_1, f_2.] For applications of theorem II see L. Cesari [Rend. Circ. Mat. Palermo 61, 225–268 (1937)] and ~~unpublished work of~~ V. Bononcini [Atti Sem. Mat. Fiz. Univ. Modena 5, 154–164 (1951); these Rev. 14, 750].

L. Cesari.

Dževaršeĭšvili, A. G. On approximation of a function of two variables by trigonometric polynomials. Soobščeniya Akad. Nauk Gruzin. SSR 13, 449–455 (1952). (Russian)

Let $F(x, y)$ be a continuous function of period 2π in each variable. By $E_{mn}[F]$ we denote the best approximation of F by trigonometric polynomials $T_{mn}(x, y)$ of order m in x and n in y. Let

$$\omega(F, \delta, \eta)=\sup|F(x_1, y_1)-F(x_2, y_1)-F(x_1, y_2)+F(x_2, y_2)|,\quad |x_1-x_2|\leqq\delta,\quad |y_1-y_2|\leqq\eta,$$

be the modulus of continuity of F in x, y, and let $\omega_{x_0}(F, \eta)$ be the modulus of continuity of F in y alone, for $x=x_0$ fixed; $\omega_{y_0}(F, \delta)$ is defined similarly. Finally, let $\Delta^2(F, x, y, h, k)$ be equal to

$$F(x+h, y+k)+F(x-h, y+k)+F(x+h, y-k)+F(x-h, y-k)+4F(x, y)-2F(x+h, y)-2F(x-h, y)-2F(x, y+k)-2F(x, y-k).$$

Then: 1) We have

$$E_{mn}(F)\leqq C[\omega(F, m^{-1}, n^{-1})+\omega_{x_0}(F, n^{-1})+\omega_{y_0}(F, m^{-1})]$$

provided there exists a constant $M>0$ and a point (x_0, y_0) such that

$$\omega_x(F, \eta)\leqq M\omega_{x_0}(F, \eta),\quad \omega_y(F, \delta)\leqq M\omega_{y_0}(F, \delta)$$

for every (x, y). 2) If $E_{mn}[F]\leqq A/mn$ $(m, n=1, 2, \cdots)$, then, with a suitable $M>0$, we have $|\Delta^2(F, x, y, h, k)|\leqq M(h+k)$. 3) If, in addition, $\lambda^{-1}\leqq h/k\leqq\lambda$, then $|\Delta^2(F, x, y, h, k)|\leqq Mhk$, with $M=\mu_\lambda$. 4) If

$$|\Delta^2(F, x, y, h, k)|\leqq Mhk,$$
$$|F(x+h, y)+F(x-h, y)-2F(x, y)|\leqq Mh,$$

and if the latter condition holds with x and y interchanged, then $E_{mn}[F]\leqq A(m^{-1}+n^{-1})$.

A. Zygmund.

Dževaršeĭšvili, A. G. On summation of double trigonometric series by Riemann's method. Soobščeniya Akad. Nauk Gruzin. SSR 13, 513–518 (1952). (Russian)

Let $\Delta^2(F, x, y, h, k)$ be defined as in the preceding review. With every double trigonometric series (S) we associate a function $F(x, y)$ obtained as the sum of the series integrated termwise twice with respect to x and twice with respect to y. The series (S) is said to be summable R_λ at the point (x, y) to sum s if the series defining F converges in the neighborhood of (x, y) and if $\lim_{u,v\to0}\Delta^2(F, x, y, 2u, 2v)/16u^2v^2=s$, provided u and v tend to $+0$ in such a fashion that $\lambda^{-1}\leqq u/v\leqq\lambda$. The main result of the paper is that if (S) is the Fourier series of an $f\in L$ and if it converges in the Pringsheim sense almost everywhere, then it is summable R_λ almost everywhere, to the same sum.

A. Zygmund.

Berger, Erich R. Comments on "The Dirac delta function and the summation of Fourier series". J. Appl. Phys. 24, 951 (1953).

See same J. 23, 906–909 (1952); these Rev. 14, 162.

de Mira Fernandes, A. Una generalizzazione della serie di Fourier. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. 2, 105–110 (1952).

The author notes the possibility of expanding a function holomorphic in a strip in a series involving powers of certain functions of period $w=re^{i\theta}$.

P. Civin (Princeton, N. J.).

Francesco-Saverio, Rossi. Sui coefficienti di Legendre di una funzione limitata, compresa fra limiti assegnati. Ann. Scuola Norm. Super. Pisa (3) 6 (1952), 317–322 (1953).

The author gives a set of conditions on the numbers a_n which are necessary and sufficient for them to be the Fourier-Legendre coefficients of a measurable $f(x)$ such that $|f(x)|\leqq1$ almost everywhere. The conditions are obtained by means of the fact that the sine coefficients b_n of

$f(\cos t)$ and the a_n are expressible in terms of each other, combined with Ghizzetti's conditions for the trigonometric moment problem [same Ann. (3) **4**, 131–156 (1950); these Rev. **12**, 94]. [Other solutions of the author's problem could be obtained from the theory of the power moment problem for a finite interval.] *R. P. Boas, Jr.* (Evanston, Ill.).

Petersen, Richard, and Skovgaard, Helge. On an equiconvergence theorem for Laguerre series. Mat. Tidsskr. B. **1952**, 14–27 (1952).

The problem of equiconvergence of the Laguerre and Fourier series has been treated by W. Rotach [Dissertation, Zürich, 1925] and by the reviewer [Math. Z. **25**, 87–115 (1926)]; the conditions used by these authors are overlapping. The authors of the present paper give a new proof of the theorem using only "elementary" arguments, avoiding, in particular, the application of the method of steepest descent. *G. Szegö* (Stanford, Calif.).

Gagua, M. B. On a theorem of S. N. Bernšteĭn. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 207–208 (1952). (Russian)

Let E be an arbitrary closed set in Euclidean n-space, and $u_0, u_1, v_1, u_2, v_2, \cdots$ be an orthonormal system of real, continuous functions on E. Designate by M_n $(n=0, 1, \cdots)$ the set of functions of the form $w_n(x)=a_0u_0+\sum_{i=1}^n(a_iu_i+b_iv_i)$ where a_0, a_i, b_i are arbitrary real constants. Let $f(x)$ be a real function defined on E, and set

$$E^n(f)=\inf_{w_n\in M_n}\sup_{x\in E}|f(x)-w_n(x)|.$$

The author announces the following generalization of a theorem of S. Bernstein [C. R. Acad. Sci. Paris **206**, 1520–1523 (1938)]. Given the numbers $A_0\geqq A_1\geqq\cdots$, $\lim_{n\to\infty}A_n=0$, there exists a continuous function $f(x)$, $x\in E$, for which $E^n(f)=A_n$ $(n=0, 1, \cdots)$. A similar theorem is stated where the functions f, u_i, v_i are solutions of an elliptic differential equation. *P. Davis.*

Szász, Otto. On closed sets of rational functions. Ann. Mat. Pura Appl. (4) **34**, 195–218 (1953).

The author proves that $T=\{1/(x^2+a_n^2)\}_{n=1}^\infty$ $(a_n>0)$ is closed in $L_2(0, \infty)$ if and only if (1) $\sum a_n/(1+|a_n|^2)=\infty$. The method of proof is to show first the closure of the set $\{1/(x^2+b^2)\}$ where b ranges over all complex numbers with positive real part and then to prove by direct calculation that any function $1/(x^2+b^2)$ can be approximated by linear combinations from T if and only if (1) is true. By similar reasoning it is shown that (1) is also necessary and sufficient for the closure in $L_2(0, \infty)$ of $\{x/(x^2+a_n^2)\}$. Other closure problems investigated are: $\{x^{\alpha+n}e^{-x/2}\}$ in $L_2(0, \infty)$; $\{(cx-a_n)/(1-\bar{c}xa_n)\}$ in $C(-1, 1)$ (closed for $|c|\leqq 1$, $|a_n|<1$, $\sum(1-|a_n|)=\infty$); $\{1/(1-a_nz)\}$, $|a_n|<1$, in the class of functions regular and bounded in $|z|<1$ (closed if and only if $\sum(1-|a_n|)=\infty$); closure of T in $C(0, \infty)$. *W. H. J. Fuchs* (Ithaca, N. Y.).

Sz.-Nagy, Béla. Über die Ungleichung von H. Bohr. Math. Nachr. **9**, 255–259 (1953).

The inequality in question is that if $f(x)$ is an almost periodic trigonometric polynomial $\sum c_k \exp(i\lambda_k x)$, with $\lambda_k\geqq\Lambda>0$, then its integral $F(x)$ with mean-value zero satisfies $|F(x)|\leqq\frac{1}{2}\pi\Lambda^{-1}\sup|f(x)|$. The author reproduces a very simple real-variable proof previously given (in Hungarian) jointly with A. Strausz [Math. Naturwiss. Anz. Ungar. Akad. Wiss. **57**, 121–135 (1938)]. *R. P. Boas, Jr.*

Kopeć, J. On vector-valued almost periodic functions. Ann. Soc. Polon. Math. **25** (1952), 100–105 (1953).

By applying Banach space theorems (notably the existence of a denumerable weak*-dense set of functionals in the conjugate space of a separable Banach space and Mazur's theorem to the effect that weak sequential limit points of convex sets are strong sequential limit points), the author establishes some of the classical results of Bochner [Acta Math. **61**, 149–184 (1933)] on vector-valued almost periodic functions. The theorems for an almost periodic $x(t)$ are: 1) $M_t\{x(t)e^{-i\lambda t}\}\neq 0$ for at most countably many λ; 2) let $S_q(\theta)=M_t\{x(t+\theta)K^q(\theta)\}$, where $K^q(\theta)$ is the classical Bochner polynomial; then $\|S_q(t)-x(t)\|\to 0$ uniformly. *B. Gelbaum* (Minneapolis, Minn.).

Kahane, Jean-Pierre. Quasi analyticité des fonctions moyenne-périodiques. C. R. Acad. Sci. Paris **236**, 569–571 (1953).

L'auteur commence par définir la "transformée de Fourier", $F(w)=\mathcal{T}(f)$ d'une fonction f moyenne-périodique (de Schwartz). $F(w)$ est le quotient de $\mathcal{T}(g)$ par $\mathcal{T}(\mu)$, où μ vérifie $f*\mu=0$ et où $g(y)=\int_{-\infty}^0=-\int_{0+}^\infty f(x)d\mu(y-x)$. L'ensemble des pôles de F est le spectre de f. $\mathcal{E}(\Lambda)$ est l'ensemble des fonctions f de spectre contenu dans la suite Λ donnée. L'auteur démontre les théorèmes suivants: Si f est bornée (le spectre est alors réel et simple, c'est-à-dire, les pôles de F sont simples), si $\liminf_{\alpha\to 0}J(\alpha)\int_0^\alpha|f|=0$, avec $\alpha^{-1}\log J(\alpha)=r(\alpha)\downarrow$, $\alpha(r(\alpha))=\alpha$ et si

$$\limsup_{r=\infty} r[\alpha(r)-2\pi\bar{D}(r)]>0,$$

où

$$\bar{D}(r)=(2r)^{-1}\int_{-r}^{r}|t^{-1}|N(t)dt,$$

$N(t)$ désignant le nombre de λ_j sur $[0, t]$, alors $f=0$. Ce résultat est très précis quand Λ est voisin de l'ensemble des entiers. Il contient un théorème de Levine et Lifschitz généralisant celui de Mandelbrojt. Un résultat très précis est aussi donné pour le cas où Λ est très lacunaire. Si

$$\liminf_{n=\infty} M_n|\lambda_1\lambda_2\cdots\lambda_{n+1}|^{-1}=0$$

toute fonction de $C\{M_n\}\cap\mathcal{E}(\Lambda)$, s'annulant avec toutes ses dérivées en un point, est identiquement nulle. Citons encore le théorème suivant: Si $g\in C\{M_n\}$ quasi analytique sur toute la droite, et si g est uniformément approchable sur un segment de longueur supérieure à leur moyenne période par des fonctions de spectre Λ, alors $g\in\mathcal{E}(\Lambda)$. *S. Mandelbrojt* (Houston, Tex.).

Hewitt, Edwin. Remarks on the inversion of Fourier-Stieltjes transforms. Ann. of Math. (2) **57**, 458–474 (1953).

Let G be a locally compact Abelian group with character group $\hat{G}$; let φ be a Radon measure on G,

$$\hat{\varphi}(x)=\int_G(x, \hat{x})d\varphi(x)$$

the Fourier-Stieltjes transform of φ; φ is uniquely determined by $\hat{\varphi}$ [Cartan and Godement, Ann. Sci. Ecole Norm. Sup. (3) **64**, 79–99 (1947); these Rev. **9**, 326]. If G is R, the real line, an explicit inversion formula is known, that of P. Lévy [C. R. Acad. Sci. Paris **175**, 854–856 (1922)], which may be written, in a generalised form, as follows: if

$f \in L_1(R)$, and is of bounded variation, then

$$\lim \int_S \hat{\varphi}(\hat{x})d\hat{x} \int_G (x, \hat{x})f(x)dx = \int_G f(u)d\varphi(u),$$

where, for Levy's formula, f is the characteristic function of an interval, and the limit is taken over sets $(-T, T)$ of the real line. The author discusses whether such formulae hold for other groups, proving the generalisation of Levy's formula, with f not necessarily a characteristic function, for the real line. In R^k the formula holds if f is the characteristic function of a rectangular interval and the S are rectangles with centres at the origin tending to infinity; in R^2, the f can be taken as characteristic functions of polygons or of circles, and the S as circles tending to infinity; at a point on the boundary of the polygon or circle the inversion formula gives a fraction of φ proportional to the angle between the tangents to the boundary on either side of the point. For R^k $(k \geqq 3)$ the corresponding formulae are not universally true. Similar formulae are discussed for toroidal groups, Boolean groups, discrete groups, and finite groups.

J. L. B. Cooper (Cardiff).

Pleijel, Åke. On a theorem of Carleman. Mat. Tidsskr. B. **1952**, 39–43 (1952).

The following theorem, representing a refinement of a result of Carleman [Åttonde Skandinaviske Matematikerkongressen, Stockholm, 1934, Ohlsson, Lund, 1935, pp. 34–44], is proved. Let $A(\lambda)$ be non-decreasing for $\lambda \geqq 0$, $A(0)=0$, and for $r>0$

$$\int_0^\infty \frac{dA(\lambda)}{\lambda+r} = \frac{P \log r}{r} + \frac{Q}{r} + O(e^{-\alpha r^{1/2}}),$$

P, Q real, $\alpha>0$. Then

$$A(x) = P \log x + Q + O(x^{-1/2} \log x), \quad x \to +\infty.$$

G. Szegö (Stanford, Calif.).

Ghizzetti, Aldo. Ricerche abeliane e tauberiane compiute nell'Istituto Nazionale per le Applicazioni del Calcolo. Ann. Mat. Pura Appl. (4) **34**, 113–132 (1953).

Si espongono le ricerche destinate ad ottenere teoremi abeliani e tauberiani, relativamente alle trasformate di Laplace ed ai coefficienti di Fourier di una funzione, eseguite dai ricercatori dell'Istituto Nazionale per le Applicazioni del Calcolo di Roma, nel periodo 1935–1952. Si segnalano nuove possibili ricerche. *Author's summary.*

Azpeitia, A. G. Note on a Laplace transform. Gaceta Mat. (1) **5**, 6–7 (1953). (Spanish)

Rodriguez-Salinas, Baltasar. On certain asymptotic developments of curvilinear Laplace integrals. Revista Mat. Hisp.-Amer. (4) **13**, 120–127 (1953). (Spanish)

The author considers the asymptotic behaviour of the integral $\int_\alpha^\beta e^{-sx^p} g(x)dx$. *I. I. Hirschman, Jr.*

Doetsch, Gustav. Asymptotic developments and the Laplace transform. Revista Mat. Hisp.-Amer. (4) **13**, 5–60 (1953). (Spanish)

This is an expository article. *I. I. Hirschman, Jr.*

Bhatnagar, K. P. On certain theorems on self-reciprocal functions. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 42–69 (1953).

The author considers the "general transform" whose kernel is the resultant [in the sense of Titchmarsh, Fourier integrals, Oxford, 1937, §8.11] of the two kernels $t^{1/2}J_\nu(t)$, $t^{-3/2}J_\mu(1/t)$. A function which is self-reciprocal is said to be $R_{\mu\nu}$. The author obtains a number of results which show how $R_{\mu\nu}$ functions may be composed of functions which are self-reciprocal in the component Hankel transformations, or how an $R_{\mu\nu}$ function can be transformed into another $R_{\mu\nu}$ function, or into an $R_{\kappa\lambda}$ function. He indicates briefly the extension of his work to kernels which are resultants of several (rather than two) Bessel function kernels.

A. Erdélyi (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Morris, J. Note on the derivation by simple algebra of Routh's stability criterion for the biquadratic characteristic equation. Quart. J. Mech. Appl. Math. **6**, 255–256 (1953).

Kelly, J. B. On factorization of polynomials. Amer. Math. Monthly **60**, 375–379 (1953).

Let $f(x)$ be a polynomial with rational integral coefficients and leading coefficient unity. To determine in a finite number of steps the factorization of $f(x)$ over the rational field, the author proposes the following procedure which is believed to require less effort than Kronecker's method. Define

$$\Delta_{j,k} = \Delta_{j,k} f(x) = \prod \sigma_j(r_1', \cdots, r_k'),$$

where σ_j denotes the elementary symmetric function of order j of $r_1', \cdots, r_k'$ and the product is over all combinations of the roots of $f(x)$ taken k at a time. Then if $f(x)=g(x)h(x)$, where $g(x)=x^k+\sum(-1)^j t_j x^{k-j}$, it is necessary that $t_j | \Delta_{j,k}$, $j=1, 2, \cdots$. A few hints for shortening the work are indicated. *L. Carlitz* (Durham, N. C.).

Bernštein, S. N. A condition necessary and sufficient for an even nondecreasing function to be a weight function. Doklady Akad. Nauk SSSR (N.S.) **88**, 589–592; correction, **90**, 124 (1953). (Russian)

The author calls $\Phi(x)$ a weight function if every $f(x)$ which is continuous on $(-\infty, \infty)$ and $o\{\Phi(x)\}$ can be approximated by polynomials in the sense that to $\epsilon>0$ there corresponds a polynomial $P(x)$ with $|f(x)-P(x)|<\epsilon\Phi(x)$, $-\infty<x<\infty$. The problem of characterizing weight functions has an extensive literature. Here the author gives conditions which are necessary and sufficient for a positive $\Phi(x)$, which is even, and nondecreasing for $x>0$, to be a weight function. These are as follows. Let $R_n(x)$ stand for any even polynomial of arbitrary degree n, such that (with a given c, $0<c<1$) $R_n(0) \geqq c\Phi(0)$ and $|R_n(x)| \leqq \Phi(x)$; let $\alpha_{k,n} \pm i\beta_{k,n}$, $\beta_{k,n}>0$, be the roots of $R_n(x)$; then the sums $\sum \beta_{k,n}/(\alpha^2_{k,n}+\beta^2_{k,n})$, extended over all the roots (other than 0) of $R_n(x)$, shall be unbounded. Equivalently, the integrals $\int_0^\infty x^{-2} \log |R_n(x)/R_n(0)| dx$ shall be unbounded above. [The problem of characterizing weight functions has been solved independently, without restrictions of evenness or monotonicity on $\Phi(x)$, by H. Pollard in a paper forthcoming in Proc. Amer. Math. Soc.; see Bull. Amer. Math. Soc. **59**, 352 (1953), abstract 376.] *R. P. Boas, Jr.*

Basone, Nelly. General recurrence equation of the polynomials orthogonal to the Pearsonian probability functions. An. Soc. Ci. Argentina **155**, 3–10 (1953). (Spanish)

In C. E. Dieulefait's work on correlation theory [cf., e.g., Biometrika **26**, 379–403 (1935)] there is presented a form

for the computation of polynomials orthogonal to the Pearsonian probability functions. This represents the generalization of the formula by which O. Rodrigues defined the Legendre polynomials. The purpose of this article is to show the validity of the general formula and to find by means of it the recurrence equation which gives the polynomials orthogonal to the probability functions of the group P. From the general formula the recurrence equations of the polynomials of Legendre, Hermite, Laguerre, and Jacoby are found. *E. Frank* (Chicago, Ill.).

Picone, Mauro. Una semplicissima formola di maggiorazione per i polinomi di Legendre e per le loro derivate. Boll. Un. Mat. Ital. (3) 8, 1–2 (1953).

For the kth derivative of $P_n(z)$ the following bound is obtained:

$$|P_n^{(k)}(z)| \leq \frac{1\cdot 3\cdots(2n-1)}{(n-k)!}(|z|+1)^{n-k}.$$

Here z is arbitrary complex. *G. Szegö* (Stanford, Calif.).

Ossicini, Alessandro. Funzione generatrice dei prodotti di due particolari polinomi di Jacobi. Boll. Un. Mat. Ital. (3) 8, 45–52 (1953).

Using the customary notation $P_n^{(\alpha,\beta)}(x)$ for the Jacobi polynomials the following generating function is derived:

$$\sum \frac{n!\Gamma(n+\nu+3/2)}{\Gamma(n+\nu+1)\Gamma(n+3/2)} P_n^{(\nu,1/2)}(x) P_n^{(\nu,1/2)}(y) z^n$$
$$= \pi^{-1} 2^{\nu+1/2} u^{-\nu-1/2} v^{-1} \left\{ Q_{\nu-1/2}\left(\frac{z+1-v}{u}\right) - Q_{\nu-1/2}\left(\frac{z+1+v}{u}\right) \right\}.$$

Here $\{z(1-x)(1-y)\}^{1/2}=u$, $\{z(1+x)(1+y)\}^{1/2}=v$, and Q designates the Legendre function of the second kind. Various special cases are considered. *G. Szegö.*

Brafman, Fred. Unusual generating functions for ultraspherical polynomials. Michigan Math. J. 1 (1952), 131–138 (1953).

Using the customary notation $P_n^{(\alpha,\beta)}(x)$ for the Jacobi polynomials, the following identity is derived:

$$\sum_{n=0}^{\infty} \frac{\Gamma(k+\alpha+1)n!}{\Gamma(n+\alpha+1)k!} P_n^{(\alpha,\alpha)}(x) z^n = 2^\alpha \rho^{-1}(1-z^2+\rho\mu)^{-\alpha}$$

where $\rho=(1-2xz+z^2)^{1/2}$, $\mu=(1+2xz+z^2)^{1/2}$, $k=[n/2]$.
G. Szegö (Stanford, Calif.).

Mohr, Ernst. Zur Theorie der Tschebyscheffschen Polynome. Ann. Scuola Norm. Super. Pisa (3) 6 (1952), 245–253 (1953).

The Tchebychev polynomial of a given degree corresponding to a given curve can be represented as a linear combination of the associated Faber polynomials. Concerning this representation P. Heuser [Math. Z. 51, 574–585 (1949); these Rev. 10, 696] has proved a theorem for which a new proof is furnished in the present note. *G. Szegö.*

Special Functions

***Buchholz, Herbert. Die konfluente hypergeometrische Funktion mit besonderer Berücksichtigung ihrer Anwendungen.** Ergebnisse der angewandten Mathematik. Bd. 2. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953. xvi+234 pp. DM 36.00.

This book differs from Tricomi's "Lezioni sulle funzioni ipergeometriche confluenti" [Gheroni, Torino, 1952; these Rev. 14, 269] to such an extent that the two works complement each other in a very useful manner. Tricomi's book is an eminently readable introduction to its subject, Buchholz' monograph contains practically all information available about confluent hypergeometric functions, often in abbreviated form and sometimes without proofs. Tricomi uses the notation derived from the theory of hypergeometric series, Buchholz the notation introduced by Whittaker. (There is a third notation, that used in nuclear physics under the name of Coulomb wave functions. This notation is mentioned in Buchholz's book, but no extensive collection of formulas seems to be available.)

The principal functions investigated in this book are $M_{k,m/2}(z)$ and $W_{k,m/2}(z)$, and they differ from the functions introduced by Whittaker only in using $m/2$ in place of Whittaker's parameter m. Frequently the function

$$\mathfrak{M}_{k,m/2}(z) = M_{k,m/2}(z)/\Gamma(m+1)$$

is used instead of M, it being an entire function of m. All these are called parabolic functions. Of special confluent hypergeometric functions, Laguerre and Hermite polynomials and parabolic cylinder functions are discussed to a considerable extent; other special cases are mentioned briefly.

In Chapter I Kummer's series, ${}_1F_1$, is introduced as a limiting case of Gauss' hypergeometric series, and the properties of this series are investigated. Then Whittaker's form of the confluent hypergeometric differential equation is given, its solutions are defined, and their basic properties are investigated. Differential equations which may be reduced to Whittaker's equation lead to Weber's equation and to parabolic cylinder functions. An inhomogeneous Whittaker equation is also studied. The separation of the wave equation in the coordinates of the parabolic cylinder and of the paraboloid of revolution, and the occurrence of parabolic functions and parabolic cylinder functions in this context is elucidated.

In Chapter II important integral representations of parabolic functions are obtained and discussed in great detail, thus preparing the ground for their manifold applications. Recurrence relations are obtained from the integral representations. A few integrals representing products of parabolic functions are also given.

Chapter III is devoted to the asymptotic behavior of parabolic functions. In separate sections the asymptotic behavior is described when one of the three quantities z, k, m is large, the others being fixed, or when k and m are both large, and their difference is fixed. Then follows the asymptotic investigation of these functions for fixed m and fixed real $z/4k$. Three cases must be distinguished according as $z/4k$ is negative, positive and <1, or >1. There is also a supplementary result for the case that $z/4k$ is near unity. Most of the asymptotic results are obtained by applying the method of steepest descents to suitable integral representations, but a brief summary of the results obtained by Langer's method is appended.

In Chapter IV integrals involving parabolic functions are evaluated. The variable of integration is mostly z, both indefinite and definite integrals are given, and the Laplace and Mellin transforms (also Hankel transforms) of parabolic functions are included. Some infinite series are also in this chapter.

Chapter V is devoted to a discussion of polynomials which are particular cases of, or are associated with, parabolic functions. Laguerre and Hermite polynomials are given in considerable detail (many series and integrals with

these polynomials being listed); Charlier polynomials, Bateman's k-function, and associated polynomials are mentioned more briefly.

Chapter VI gives the expression, as infinite series of products of parabolic functions, or as integrals over the parameter k of such functions, of several types of wave functions. The results of this chapter are of very great importance for the application of confluent hypergeometric functions in wave problems, and much of the material presented here is the author's own work.

In Chapter VII the zeros of parabolic functions are investigated, as k and m are fixed and z varies (both for M and W), and as m and z are fixed and k varies (for M). These investigations are used to illustrate the application of parabolic functions in eigenvalue problems. The examples chosen are the vibrations of an inhomogeneous string, and Green's function for the wave equation in a region bounded by confocal paraboloids of revolution.

Appendix I is a very useful summary of all particular cases of parabolic functions, and gives also information about numerical tables of these functions. Appendix II is an extensive bibliography. An index of notations and symbols at the beginning of the book and a subject index at the end are of considerable help in using this book as a valuable work of reference, certainly the most complete one on its subject. *A. Erdélyi* (Pasadena, Calif.).

***Lense, Josef. Reihenentwicklungen in der mathematischen Physik.** 3te Aufl. Walter de Gruyter & Co., Berlin, 1953. 216 pp. DM 26.

For a detailed review of the 2d edition of 1947 see these Rev. 11, 105. "Die dritte Auflage unterscheidet sich nicht wesentlich von der zweiten. Es wurde die Gelegenheit benützt, sämtliche bekannt gewordenen Druck- und Schreibfehler zu verbessern. Ferner wurden neben kleinen Textänderungen manche Beweise vereinfacht und einige Ziffern umgestellt".

Blanuša, Danilo. A type of integral theorems for Bessel functions. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **277**, 5–128 (1950). (Serbo-Croatian)

This paper discusses the Laplace convolution of

$$t^m \Lambda_n (c(t^2-a^2)^{1/2})$$

with other functions of the same form, where Λ_n is the entire function occurring in the definition of Bessel functions of the first kind (notation as in Jahnke-Emde). A great many recurrence and other relations are obtained. The work is based partly on previous results by the author [same Rad 271, 83–143 (1948); these Rev. 11, 245]. *A. Erdélyi.*

Meijer, C. S. Expansion theorems for the G-function. IV. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 187–193 (1953).

[For parts I to III see same Proc. **55**, 369–379, 483–487 (1952); **56**, 43–49 (1953); these Rev. **14**, 469, 642, 748.] The present part contains a proof of the "second expansion theorem"

$$\left[\prod_{j=1}^{l}\Gamma(d_j)\right] G^{m+l,n}_{p+l,q+l}\left(\lambda w\,\middle|\,\begin{matrix} a_r, c_j \\ d_j, b_s\end{matrix}\right)$$
$$=\left[\prod_{j=1}^{l}\Gamma(c_j)\right]^{-1}\sum_{k=0}^{\infty}\frac{1}{k!}\,{}_{l+1}F_l\left(-k, d_j; c_j; \frac{1}{\lambda}\right)$$
$$\times G^{m,n+1}_{p+1,q+1}\left(\omega\,\middle|\,\begin{matrix} 1-k, & a_r \\ b_s, & 1\end{matrix}\right)$$

under several sets of conditions of validity, together with auxiliary formulas. *A. Erdélyi* (Pasadena, Calif.).

Vroelant, Claude. Calcul des intégrales intervenant pour certaines formes approchées de la fonction d'onde. C. R. Acad. Sci. Paris **236**, 2504–2506 (1953).

Il est donné dans cette note le principe du calcul des diverses intégrales intervenant dans les calculs moléculaires pour des fonctions d'onde approchées de la forme

$$\Psi=\sum_k f_k(\cdots, x_i, \cdots)e^{-g_k(\cdots,x_i,\cdots)},$$

les f_k étant des polynomes et les g_k étant des polynomes du second degré assujetti à certaines conditions.

Author's summary.

Harmonic Functions, Potential Theory

Myrberg, Lauri. Über die Existenz von positiven harmonischen Funktionen auf Riemannschen Flächen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **146**, 6 pp. (1953).

It is shown that if the Green's function $g(P, Q)$ does not tend to zero on some sequence $\{P_n\}$ approaching the boundary, then a subsequence of $\{g(P, P_n)\}$ tends to a positive harmonic function. *L. Sario* (Cambridge, Mass.).

Allen, A. C. On positive harmonic functions. Proc. Cambridge Philos. Soc. **48**, 571–577 (1952).

Any function h positive harmonic in the upper half, $y>0$, of the $z=x+iy$ plane can be represented by the Poisson-Stieltjes integral formula

$$h(z)=cy+\int_{-\infty}^{\infty}\frac{y(\xi^2+1)}{y^2+(x-\xi)^2}dg(\xi)=(c, g; z)$$

where $c\geqq 0$ and g is bounded nondecreasing. Let $f: z\rightarrow Z$ be a simple conformal mapping with inverse F from a portion δ of the upper half, $y>0$, of the z-plane onto the upper half, $Y>0$, of the $Z=X+iY$ plane such that an open interval (a, b) of the x-axis is carried by continuous extension of f into the X-axis. The function $H(Z)=h(F(Z))$ is then positive harmonic in $Y>0$, and so has the representation $H(Z)=(C, G; Z)$ for suitable C, G. The problem of relation of G to g for the particular mapping $Z=z^\lambda$ $(\lambda>1)$ was mentioned by Loomis and solved by Verblunsky [same Proc. **44**, 289–291 (1948); these Rev. **10**, 296]. The author here solves the general case. Reflect the mapping $z\rightarrow Z$ in the x, X-axes, whereupon $X=f(x)$ is seen to be regular and monotonic (increasing, say) on the interval (a, b) with monotonic inverse $x=F(X)$. The G difference between A and B where $a<F(A)<F(B)<b$ is given by the formula

$$G(B)-G(A)=\int_{F(A)}^{F(B)}\frac{\xi^2+1}{f(\xi)^2+1}f'(\xi)dg(\xi),$$

appropriately unilaterally interpreted at g jumps.

W. Gustin (Bloomington, Ind.).

Sibagaki, Wasao, and Ono, Akira. On the mean-value theorem of harmonic functions. Mem. Fac. Sci. Kyūsyū Univ. A. **7**, 41–50 (1952).

Pour que f mesurable soit solution généralisée de l'équation de Laplace (donc presque partout égale à une fonction harmonique ordinaire) il faut et il suffit qu'en presque tout point x, f soit égale à sa valeur moyenne sur toute boule de

centre x. Tout ceci est dans L. Schwartz [Théorie des distributions, t. I, II, Hermann, Paris, 1950–51; ces Rev. 12, 31, 833]. *J. Deny* (Strasbourg).

Pucci, Carlo. Bounds for solutions of Laplace's equation satisfying mixed conditions. J. Rational Mech. Anal. 2, 299–302 (1953).

Remarques sur certaines majorantes et minorantes pour une fonction u satisfaisant à $\Delta u=f$ dans un domaine D et à une condition à la frontière du type $\alpha du/dl+\beta u=\gamma$, l étant une direction variable pénétrant dans D, α, β, γ des fonctions données soumises à quelques restrictions. Une application numérique est faite dans le cas d'un problème mixte pour le cercle. *J. Deny* (Strasbourg).

Wolska, J. Le problème aux limites de H. Poincaré pour le système de fonctions. Prace Mat.-Fiz. 48, 67–78 (1952).

The problem solved is that of finding a set of functions $u_1(x, y), \cdots, u_m(x, y)$ which in a domain D, bounded by a regular analytic curve C, satisfy the equation of Laplace, while on C one has:

$$\frac{du_\alpha}{dn}+\sum_{\beta=1}^{m}\left(a_{\alpha\beta}(s)u_\beta+b_{\alpha\beta}(s)\frac{du_\beta}{ds}\right)+f_\alpha(s)=0 \quad (\alpha=1, \cdots, m)$$

where s is the length of arc on C. The coefficients are analytic, of period equal to the length of C; they are real for s real and have no singularities in an infinite band containing the axis of reals. The solutions are obtained in the form of logarithmic potentials of masses distributed on C, in accord with densities $\mu_\alpha(\sigma)$, which are to be found. The $\mu_\alpha(\sigma)$ are shown to satisfy a system of integral equations in the sense of principal values, with kernels having a logarithmic pole. An iteration with the aid of Poincaré's and Pogorzelski's [Ann. Acad. Polonaise Sci. Tech. Varsovie 1, 82–114 (1935); Math. Z. 44, 427–444 (1938)] formulas yields a system with kernels bounded in the ordinary sense. It is then shown that the μ_α satisfy a system of the form

$$\mu_\alpha(s)=f_\alpha'(s)+\sum_{\gamma}^{m}\int_0^a\Phi_{\alpha\gamma}(s, \tau)\mu_\gamma(\tau)d\tau+\sum_{\gamma}^{m}\int_s^a\Psi_{\alpha\gamma}(s, \tau)\mu_\gamma(\tau)d\tau$$

($\alpha=1, \cdots, m$), where the kernels are bounded, analytic for s, τ in a neighborhood of the segment $[0, a]$. Resolution of a certain system of Volterra equations gives a Fredholm system, which is finally transformed into a single Fredholm equation. The densities μ_α arise from the solution of the latter equation and they are shown to be analytic in a neighborhood of the segment $[0, a]$. A special case of the problem solved by the author is found in Poincaré's Leçons de mécanique céleste, t. III [Gauthier-Villars, Paris, 1910]. *W. J. Trjitzinsky* (Urbana, Ill.).

Mitrović, Dragiša. Une remarque sur l'intégral de Dirichlet. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 8, 44–46 (1953). (Serbo-Croatian. French summary)

Frostman, Otto. Distributions de masses normées par la métrique de L^p. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 90–100 (1952).

This paper appeared earlier in Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 21, no. 13 (1951); these Rev. 13, 942.

Aissen, Michael. A class of super-additive functions. Proc. Amer. Math. Soc. 4, 360–362 (1953).

The stress function $u(p; E)=u(E)$ of an open set E with analytic boundary is a solution of $\nabla^2u=-2$ satisfying on the boundary the condition $u=0$. The following property of this function is proved. Let E_ν, E be bounded open sets whose components are simply-connected sets bounded by a finite number of analytic arcs; let $E_\nu\subset E$, $\nu=1, 2, \cdots, n$. The following inequality holds:

$$u(E)\geqq\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\leqq m_1<\cdots<m_k\leqq n}u(E_{m_1\cdots m_k})$$

where the last set is defined by $\bigcap_{\nu=1}^{k}E_{m_\nu}$. *G. Szegö.*

***Diaz, J. B. On the estimation of torsional rigidity and other physical quantities.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 259–263. The American Society of Mechanical Engineers, New York, N. Y., 1952.

In order to establish bounds for the torsional rigidity of a simply or multiply connected domain D bounded by the system of curves $C=\sum C_i$, the author develops certain general methods to obtain bounds for the Dirichlet integral (*) $\int_D(\nabla\phi)^2d\sigma$ where ϕ satisfies the conditions $\nabla^2\phi=0$ in D, $\partial\phi/\partial n=\partial f/\partial s$ on C. Let w_k be arbitrary functions in D for which the determinant $\int_D\nabla w_i\cdot\nabla w_k d\sigma$, $i, k=1, 2, \cdots, p$, is not zero. Then $\sum_{i=1}d_i\int_C(w_i\partial f/\partial s)ds$ is a lower bound of (*) provided

$$\sum_{j=1}^{p}d_j\int_D\nabla w_i\cdot\nabla w_j d\sigma=\int_C w_i\frac{\partial\phi}{\partial s}ds.$$

On the other hand, let $z=f+K_i$ on C_i (K_i arbitrary real constants) and $z_i=0$ on C; moreover, let the determinant $\int_D\nabla z_i\cdot\nabla z_k d\sigma$, $i, k=1, 2, \cdots, q$, be not zero. Then

$$\int_D(\nabla z)^2d\sigma-\sum_{i=1}^{q}e_i\int_D\nabla z\cdot\nabla z_i d\sigma$$

is an upper bound for (*) provided

$$\sum_{j=1}^{q}e_j\int_D\nabla z_i\cdot\nabla z_j d\sigma=\int_D\nabla z\cdot\nabla z_i d\sigma.$$

As an application it is shown that the torsional rigidity of a rectangular cross section of sides $2a$, $2b$ ($a<b$) can not exceed the quantity

$$(8/3-56/135)(b^4-a^4).$$

This result is an improvement of previous inequalities. *G. Szegö* (Stanford, Calif.).

Block, H. D. Laws of attraction having a certain generalized Newtonian property. J. Math. Physics 31, 151–153 (1952).

L'A. prova che le seguenti espressioni

$$c_1\frac{e^{-\beta r}}{r}+c_2\frac{e^{\beta r}}{r}, \quad \frac{c_1}{r}+c_2, \quad c_1\frac{\operatorname{sen}\beta r}{r}+c_2\frac{\cos\beta r}{r}$$

sono le uniche possibili funzioni potenziali $V(r)$, dotate per $r>0$ di derivate seconde continue, a tali che l'energia di interazione W fra due sfere, a densità uniforme e di masse m_1, m_2, sia

$$W(R, a, b)=m_1m_2\phi(a, b)V(R)$$

ove a e b sono i raggi delle due sfere ed R la distanza dei due centri. Risultano così completati precedenti risultati di C. J. Bouwkamp [Physica 13, 501–507 (1947); questi Rev. 9,

284] e di I. N. Sneddon e C. K. Thornhill [Proc. Cambridge Philos. Soc. **45**, 318–320 (1949)]. *C. Pucci* (Roma).

Zitarosa, Antonio. Su un problema misto di Dirichlet-Neumann. Ricerche Mat. **1**, 255–286 (1952).

The problem considered is the mixed boundary-value problem for Laplace's equation inside a dihedral angle in space, the value of the function being given on one boundary half-plane and that of the normal derivative on the other. Two uniqueness theorems are given, a definite integral formula for the solution determined, if the solution exists, and an existence theorem proved for the case the dihedral angle degenerates into a half space. Even the statements of the theorems are too complex to give in this review; for instance, the first uniqueness theorem requires one complete large page to state. *J. W. Green* (Los Angeles, Calif.).

Differential Equations

Hukuhara, Masuo. Sur un théorème de Kneser. J. Fac. Sci. Univ. Tokyo. Sect. I. **6**, 329–344 (1953).

Consider the system of ordinary differential equations

$$(1)\qquad \frac{dy_i}{dx}=f_i(x, y_1, y_2, \cdots, y_n)\quad (i=1, 2, \cdots, n).$$

Let Ω be the set defined by

$$(2)\qquad a\leqq x<a', \quad \sum_{i=1}^{n} y_i^2<\infty$$

and E a set which is closed on Ω. Let A be a continuum lying in E, and $R(A)$ the region covered by the integral curves of (1) going through points of A. Finally, denote by $S_\xi(A)$ the intersection of $R(A)$ with the plane $x=\xi$ $(a\leqq\xi<a')$. Definition: a family of solutions of (1) has the Kneser property (at right) if for ξ at the right of A, S_ξ is a continuum.

Under the assumption that the right members of (1) are bounded and continuous in E, the author obtains sufficient conditions (which are too complicated to be reproduced here) in order that the family of solutions of (1) have the Kneser property (at the right). He thus generalizes a result of H. Kneser [S.-B. Preuss. Akad. Wiss. **1923**, 171–174] who dealt with the case where E is the closure of Ω.

E. H. Rothe (Ann Arbor, Mich.).

Krein, M. G. On the theory of entire matrix functions of exponential type. Ukrain. Mat. Žurnal **3**, 164–173 (1951). (Russian)

An entire matrix function of exponential type is a square matrix of order n whose elements are entire functions of exponential type. The author takes the norm $\|A\|$ of the matrix A to be the norm of the associated linear transformation in n-space. His main theorem is a generalization of a theorem he previously proved in the case $n=1$ [Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 309–326 (1947); these Rev. **9**, 179]: a matrix function $F(z)$ which admits the representation

$$F^{-1}(z)=C_{-1}z^{-1}+C_0+C_1z+\cdots+C_{p-1}z^{p-1}+z^p\sum_{k=1}^{\infty}A_k/(z-\alpha_k),$$

with real α_k and $\sum\|A_k\|/|\alpha_k|<\infty$, is of exponential type and $\int_{-\infty}^{\infty}(1+x^2)^{-1}\log^+\|F(x)\|dx<\infty$.

This theorem has the following application to systems of differential equations. Let $\mathfrak{J}$ be a nonsingular constant Hermitian matrix, $H(t)$ a Hermitian matrix function whose elements are integrable in $(0, \omega)$, and consider the system (1) $dx/dt=i\lambda\mathfrak{J}H(t)x$, and the matrix equation

$$dU/dt=i\lambda\mathfrak{J}H(t)U, \quad U(0)=1.$$

The matrix $U(\omega, \lambda)$ satisfying (2) is an entire function of λ. If H is nonnegative Hermitian, the author shows that $U(\omega, \lambda)$ satisfies the conditions of his theorem on entire functions and hence has the properties asserted in that theorem. The case $n=2$ is discussed in greater detail.

R. P. Boas, Jr. (Evanston, Ill.).

Thomas, Johannes. Über gewisse lineare Differentialgleichungssysteme mit periodischen Koeffizienten. Math. Nachr. **9**, 197–200 (1953).

Let $i, j, k=1, \cdots, n$, and consider the (Jacobi) system

$$\dot{x}_i=\sum_k[b_{ik}(t)x_k+c_{ki}(t)y_k], \quad \dot{y}_i=-\sum_k[a_{ik}(t)x_k+b_{ik}(t)y_k]$$

where a_{ik}, b_{ik}, c_{ik} are continuous and p-periodic, a_{ik} and c_{ik} are symmetric and even, and b_{ik} is odd. Let $x_i^j(t)$, $y_i^j(t)$ be n solutions with $x_i^j(0)=\delta_i^j$ and $y_i^j(0)=0$, and let $x_i^{n+j}(t)$, $y_i^{n+j}(t)$ be n solutions with $x_i^{n+j}(0)=0$ and $y_i^{n+j}(0)=\delta_i^j$. Then the values of these $4n^2$ functions are connected, when $t=p$, by the following $n(2n-1)$ equations:

$$x_i^j(p)=y_j^{n+i}(p), \quad x_i^{n+\nu}(p)=x_\nu^{n+i}(p), \quad y_i^\nu(p)=y_\nu^i(p) \qquad (\nu=2, \cdots, n;\ i<\nu).$$

F. A. Ficken (Knoxville, Tenn.).

Grabar', M. I. Transformation of dynamical systems into systems of solutions of differential equations. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1952**, no. 3, 3–8 (1952). (Russian)

The author proves that every dynamical system (flow) in a separable locally compact space can be topologically imbedded in a Hilbert space H in such a fashion that the flow is defined by a differential equation $\dot{x}=F(x)$ in H. Here x is an element of H and $\dot{x}$ is defined as the strong limit, as h tends to 0, of the ratio of $x(t+h)-x(t)$ to h. This result extends previous work of the author [Doklady Akad. Nauk. SSSR (N.S.) **61**, 433–436 (1948); these Rev. **10**, 309]. As a corollary the existence of a local cross-section for the flow is established. *W. Kaplan* (Zurich).

Coddington, E. A., and Levinson, N. Perturbations of linear systems with constant coefficients possessing periodic solutions. Contributions to the Theory of Nonlinear Oscillations, vol. II, pp. 19–35. Princeton University Press, Princeton, 1952.

Consider the system (1) $\dot{x}=Ax+\mu f(x, t, \mu)$, A being a real constant matrix, μ real and small, f of period 2π in t. It is well known that (1) has, for small μ, a solution of period 2π provided that (2) $\dot{x}=Ax$ has no solution $(\neq 0)$ of period 2π. The authors discuss the "singular" case when (2) has solutions of period 2π, and find sufficient conditions for (1) to have a solution of period 2π. These conditions include the nonvanishing of a certain Jacobian, whose evaluation involves a knowledge of the real canonical form of A and the calculation of $\int_0^{2\pi}e^{tA}f(e^{-tA}c, -t, 0)dt$ as a function of the vector c. It is further shown, when $f(x, t, \mu)$ is analytic in (x, μ), that the periodic solution $p(t, \mu)$ (if it exists) is analytic in μ, and that its expansion $p(t, \mu)=\sum_0^\infty p_n(t)\mu^n$ can be found by recursively solving linear equations of the type $\dot{p}_n=Ap_n+R_n(p_0, p_1, \cdots, p_{n-1}, t)$. A similar investigation is

made when f in (1) is independent of t, and a solution of period $2\pi+O(\mu)$ is required. The "singular" case now occurs when (2) has several independent solutions of period 2π. The results obtained by the authors represent considerable extensions of what was already known for the equation $\ddot{y}+y=\mu g(y,\dot{y},t,\mu)$ [cf. J. J. Stoker, Nonlinear vibrations . . . , Interscience, New York, 1950; these Rev. 11, 666]. *G. E. H. Reuter* (Manchester).

Obi, Chike. A non-linear differential equation of the second order with periodic solutions whose associated limit cycles are algebraic curves. J. London Math. Soc. **28**, 356–360 (1953).

Starting with (1) $\dot{x}=f^{2}(x)+f(x)g(x)$, the author derives

$$(2)\qquad \ddot{x}-(\tfrac{3}{2}f'g+fg')\dot{x}-\tfrac{1}{2}f'(1-fg^{2})=0$$

as a differential equation having (1) as a particular first integral. Under suitable assumptions on $f(x)$ and $g(x)$, the phase plane trajectory of (2) obtained by setting $y=\dot{x}$ and rationalizing (1) has a branch which is a simple closed curve, and (2) possesses a periodic solution corresponding to this curve. When $f(x)$ and $g(x)$ are algebraic, this curve is algebraic, and it is thus possible for $\ddot{x}+\phi(x)\dot{x}+\psi(x)=0$ ($\phi(x)\neq 0$) to have a limit cycle which is algebraic. The author gives an explicit formula for the periodic solution of (2) in the case $f=1-x^2$, $g=\epsilon x$. However, the formula, as given, serves to define the periodic solution only over a part of the period, additional analytic expressions being needed for the remaining portion. *C. E. Langenhop.*

Amerio, Luigi. Analisi delle nozioni di "nodo", "nodo a stella" e "fuoco", estese ai sistemi di due equazioni differenziali in tre variabili. Rivista Mat. Univ. Parma **3**, 207–231 (1952).

This paper is concerned with a system of equations which is reducible, by a transformation of the dependent variables, to the form

$$dx/dt=Ax+By+\varphi(t,x,y),$$
$$dy/dt=Cx+Dy+\psi(t,x,y).$$

A, B, C, D are constants; φ, ψ are continuous and bounded as functions of t, and analytic as functions of (x,y). These functions satisfy the conditions

$$\varphi(t,0,0)=\varphi_x(t,0,0)=\varphi_y(t,0,0)=0,$$
$$\psi(t,0,0)=\psi_x(t,0,0)=\psi_y(t,0,0)=0.$$

The purpose of the author is to discuss the structure of the family of solutions in the neighborhood of the solution $x=y=0$, in terms of concepts which are suggested by Poincaré's theory of the qualitative properties of the solutions of a system of the form

$$(1)\qquad dx/dt=X(x,y),\quad dy/dt=Y(x,y).$$

[For other similar work by the author, see Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 10, 206–212, 289–297 (1951); these Rev. 12, 827; 13, 346.]

The present paper deals mainly with the case in which $A=D$ and $B=C=0$ (case of a star node). It is shown that if the solutions are represented by curves in the txy-space, ∞^1 of these curves lie on each of ∞^1 surfaces which pass through the t-axis with distinct tangent planes. The asymptotic behavior of the solutions is determined, and it is shown that the system of differential equations can be reduced to the canonical form

$$du/dt=[\lambda+F(t,u,v)]u,\quad dv/dt=[\lambda+F(t,u,v)]v$$

with λ a constant. The case in which $A=D$ and $B=-C\neq 0$ (case of a focus) is also considered, and it is shown that the transformation

$$x=u\cos Bt+v\sin Bt,\quad y=-u\sin Bt+v\cos Bt$$

reduces this to the case of a star node.

Applications of some of the results to the study of solutions of a system of the form (1) in the neighborhood of a limit cycle are indicated. *L. A. MacColl.*

de Castro Brzezicki, A. On the systems of differential equations of nonlinear mechanics. II. Revista Mat. Hisp.-Amer. (4) **12**, 317–329 (1952). (Spanish)

[For part I see same Revista (4) 12, 266–280 (1952); these Rev. 14, 754.] The author considers a system of equations of the form

$$\dot{x}=yg(t)-\varphi(x,t),\quad \dot{y}=-xg(t)+f(y,t),$$

where the functions f, g, φ are periodic with respect to t, with the common period λ. These functions are assumed to satisfy a certain complicated set of conditions, of which the following are the more noteworthy: there exist constants g_1, g_2 such that $0<g_1\leqq g(t)\leqq g_2$; $f(0,t)\neq 0$; $\partial f/\partial y\leqq 0$; there exists a positive constant b such that $f<b$; $\varphi(0,t)\neq 0$; $\partial\varphi/\partial x\geqq 0$; $\varphi(x,t)>\varphi_1(x)$, where $\varphi_1(x)$ is an increasing function such that $x\varphi_1(x)>0$, $\lim_{x\to\infty}\varphi_1(x)=\infty$. It is shown, by an application of the Brouwer fixed point theorem, that the system of equations possesses a solution which is periodic with the period λ. *L. A. MacColl* (New York, N. Y.).

Hearon, John Z. The kinetics of linear systems with special reference to periodic reactions. Bull. Math. Biophys. **15**, 121–141 (1953).

It is shown on the basis of (1) conservation of mass, (2) positive concentrations, and (3) the principle of detail balancing that periodic reactions cannot occur in a closed system described by linear differential equations. The matrix A of the rate equations must be such that $|A|=0$, $a_{ij}>0$ for $i\neq j$, $a_{ii}<0$, and $VAV^{-1}=B$, where V is diagonal and B is symmetric. These properties of A imply that the latent roots are real and non-positive and that neither catalysis nor inhibition can be described by linear equations. It is further shown that periodic reactions cannot occur in an open system for which the matrix associated with the chemical reactions has the above properties and in which the simple law of diffusion is obeyed. (From the author's abstract.) *A. S. Householder* (Oak Ridge, Tenn.).

291-314

Caligo, Domenico. Sulla integrazione delle equazioni differenziali del secondo ordine a riferimento razionale. Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, ~~322–337~~ (1952) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **353** (1953).

The author discusses the solution of the differential equation $\Phi(\ddot{x},\dot{x},x)=0$, where the left-hand member is a homogeneous polynomial in the three arguments, by a method which can be described briefly as follows. If $\dot{x}/x$ and $\ddot{x}/x$ are considered as non-homogeneous Cartesian coordinates of a point in a plane, the equation represents a certain curve C. A parametric representation of C is obtained. Then the independent variable is determined in terms of the parameter by a quadrature, and the solution is completed by another quadrature. The present note is devoted chiefly to a detailed discussion of a large number of examples, particularly ones in which the parametric representation of C is rational. *L. A. MacColl* (New York, N. Y.).

Popov, B. S. Sur une équation différentielle. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 179–182 (1953).

A theorem of Mitrinovitch is used to find the conditions under which the equation

$$(*)\qquad y''+(ae^{x}+b)y'+(Ae^{2x}+Be^{x}+C)y=0$$

has polynomial solutions in e^{x}. The author does not point out that (*) may be transformed by $z=e^{x}$ to

$$z^2d^2y/dz^2+z(az+b+1)dy/dz+(Az^2+Bz+C)y=0,$$

which has a regular singular point at the origin and may be treated by elementary methods. *E. Pinney.*

Grant, Robert B. Operational analysis of nonlinear systems. Texas J. Sci. **5**, 198–203 (1953).

Plainevaux, J. E. Sur les équations différentielles linéaires à coefficients constants. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 1142–1148 (1952).

The author solves the nonhomogeneous linear differential equation of order n with constant coefficients by factoring the differential operator, and notes that two of Heaviside's formulas follow from the result. *R. V. Churchill.*

McEwen, W. H. Spectral theory and its application to differential eigenvalue problems. Amer. Math. Monthly **60**, 223–233 (1953).

Short expository paper to be read as an introduction into the subject. *L. Gårding* (Lund).

Brewster, J. Pendleton. A modified initial condition for Cauchy's existence theorem. Proc. Amer. Math. Soc. **4**, 296–302 (1953).

If f is suitably holomorphic, the differential equation $\partial u/\partial x=f(x,y,u,\partial u/\partial y)$ has, by Cauchy's existence theorem, a unique solution u in a neighborhood of the origin satisfying $u(0,y)=c(y)$ with holomorphic c. The initial condition means that the derivatives $u_{0,n}$ (where $u_{j,k}=\partial^{j+k}u/\partial x^j\partial y^k$) are given at the origin. It is shown that under suitable assumptions Cauchy's theorem still holds if the initial condition is modified so that a sequence of derivatives $u_{k_n,n-k_n}$ $(0\leqq k_n\leqq n)$ are given at the origin. *L. Gårding* (Lund).

Feller, William. Two singular diffusion problems. Ann. of Math. (2) **54**, 173–182 (1951).

The main results of this paper concern the singular diffusion equation

$$u_t=(axu)_{xx}-((bx+c)u)_x,\quad 0<x<\infty,$$

with a, b, c constants, $a>0$. By an adroit use of Laplace transforms (on x!) the following results are obtained. (1) If $c\leqq 0$, the initial values $u(0,x)$ can be prescribed arbitrarily and they uniquely determine a solution (note that no boundary condition at $x=0$ can be imposed). The solution is positivity preserving and norm decreasing. (2) If $0<c<a$, there exists a "reflecting barrier" solution to the initial-value problem, i.e., a positivity and norm preserving solution defined by the condition that the flux at $x=0$ is 0. Moreover, there exist infinitely many positivity preserving but norm decreasing solutions among which there is only one satisfying the condition $u(t,0)<\infty$. (3) If $c>a$, there exists a positivity and norm preserving solution such that both it and its flux vanish at $x=0$. Similar methods are also applicable to the treatment of a modified Ehrenfest model of diffusion. *M. Kac* (Ithaca, N. Y.).

Rozenberg, M. D. On a nonlinear system of partial differential equations having application in the theory of filtration. Doklady Akad. Nauk SSSR (N.S.) **89**, 233–236 (1953). (Russian)

The nonlinear system of the form

$$(*)\qquad \sum_{i=1}^{n}\frac{\partial^{l_{ij}}}{\partial x^{l_{ij}}}\left[x^{m_j}f_{ij}(u_1,\cdots,u_n)\left(\frac{\partial^{s_{ij}}u_i}{\partial x^{s_{ij}}}\right)^{k_{ij}}\right]$$

$$=x^{m_j}\left[\frac{\partial}{\partial t}\varphi_j(u_1,\cdots,u_n)\right]^{\sigma_j}\quad (j=1,2,\cdots,n)$$

where $\sigma_j/(k_{ij}s_{ij}+l_{ij})=\text{constant}=\alpha$ $(i,j=1,2,\cdots,n)$ is shown to be transformable to a system of ordinary differential equations by the transformation $\eta=xt^{-\alpha}$. Taking $l_{ij}=1$, $s_{ij}=1$, $n=2$, $u_1=p$, $u_2=\rho$, $m_j=m$ and f_{ij}, ϕ_j as known functions of the saturation ρ and pressure p, the above system becomes the filtration equations which differ from the usual formulation in that f_{ij}, ϕ_j are dependent upon pressure as well as saturation. It is pointed out that the cases $m=0$, 1, 2, correspond to one-dimensional, plane-radial, and spherical-radial filtration, respectively. A three phase problem is also given as a special case of (*). *C. G. Maple* (Ames, Iowa).

Vodička, V. Conduction de la chaleur dans une barre formée de plusieurs parties en matériaux différents. Prace Mat.-Fiz. **48**, 45–52 (1952).

The problem of the conduction of heat in a thin bar with constant cross-section which consists of n parts of different materials is expressed by the following mathematical system:

$$(1)\qquad \frac{\partial u_i}{\partial t}=a_i^2\frac{\partial^2 u_i}{\partial x^2},\quad a_i^2=\frac{\lambda_i}{c_i\gamma_i},\quad x_{i-1}\leqq x\leqq x_i,\quad t>0,\quad i=1,2,\cdots,n,$$

$$(2)\qquad \frac{\partial u_1}{\partial x}-h_0u_1=0,\ x=0,\quad \frac{\partial u_n}{\partial x}+h_nu_n=0,\ x=x_n,$$

$$(3)\qquad u_i(x_i,t)=u_{i+1}(x_i,t),\quad i=1,2,\cdots,n-1,$$

$$(4)\qquad \lambda_i\frac{\partial u_i}{\partial x}=\lambda_{i+1}\frac{\partial u_{i+1}}{\partial x},\quad x=x_i,\quad i=1,2,\cdots,n-1,$$

$$(5)\qquad u_i(x,0)=f_i(x),\quad x_{i-1}\leqq x\leqq x_i,\quad i=1,2,\cdots,n,$$

where $u_i=u_i(x,t)$ is the temperature distribution in the ith part and λ_i, c_i, γ_i are the coefficient of conductibility, the specific heat, and the mass per unit volume, respectively, in the ith part of the bar. It is assumed that the initial distribution is given by $f_i(x)$ where $f_i(x)$ $(i=1,2,\cdots,n)$ are bounded integrable functions.

Using the Bernoulli method of separation of variables, the author finds particular solutions of the problem in the ith part of the bar of the form

$$(6)\qquad v_{i\nu}(x,t)=X_{i\nu}(x)T_{i\nu}(t),\quad x_{i-1}\leqq x\leqq x_i,\quad t\geqq 0,\quad i=1,2,\cdots,n,$$

which satisfy the differential equation (1) and the conditions (2)–(4). It is then shown that the functions $X_{i\nu}$ $(\nu=1,2,\cdots)$ form an orthogonal set, that is,

$$(7)\qquad \sum_{i=1}^{n}\frac{\lambda_i}{a_i^2}\int_{x_{i-1}}^{x_i}X_{i\mu}(x)X_{i\nu}(x)dx=0,\quad \mu\neq\nu,\quad \mu,\nu=1,2,\cdots,$$

so that the functions $f_i(x)$ may be expressed in the form

$$(8)\qquad f_i(x)=\sum_{\nu=1}^{\infty}c_\nu X_{i\nu}(x),\quad x_{i-1}\leqq x\leqq x_i,\quad i=1,2,\cdots,n,$$

where the coefficients are found by use of relations (7). This permits the distribution functions u_i to be expressed in terms of an infinite series of particular solutions of the form (6). For the particular case $n=1$, this reduces to the usual well-known expression for $u(x, t)$. *C. G. Maple.*

Voskresenskiĭ, K. D. On the computation of the heat regime of a shaft. Doklady Akad. Nauk SSSR (N.S.) 88, 61–62 (1953). (Russian)

The transient radial temperature variation in a cylindrical body initially at uniform temperature and having a surface thermal resistance is developed explicitly in terms of the error function. *N. A. Hall* (Minneapolis, Minn.).

Brousse, Pierre. Sur un problème de Dirichlet singulier. C. R. Acad. Sci. Paris 236, 1731–1732 (1953).

The Dirichlet problem is discussed for the equation

$$V_{xx}+V_{yy}-\frac{k}{y}V_y=0 \quad (k \text{ a positive constant})$$

when the domain contains a segment of the x-axis as part of its boundary. For continuous boundary values the Dirichlet problem is uniquely solvable. On the other hand, for the equation

$$U_{xx}+U_{yy}+\frac{k+2}{y}U_y=0,$$

obtainable from the preceding one by the transformation $V=y^{k+1}U$, the Dirichlet problem does not in general have a solution. The behavior of the solution as the singular line is approached is also discussed. *M. H. Protter.*

Garabedian, P. R., and Schiffer, M. Variational problems in the theory of elliptic partial differential equations. J. Rational Mech. Anal. 2, 137–171 (1953).

The authors present a widely applicable method for establishing the existence of analytic extremal curves in variational problems involving linear elliptic differential equations with analytic coefficients in two independent variables. They treat in detail some special variational problems for the eigenvalues of the vibrating membrane, one of which, for example, is to determine the shape of the membrane for which $L^2\lambda_2$ is a minimum, where L is the length of the boundary and λ_2 the second eigenvalue (the solution of the problem $L^2\lambda_1=\min$ is known to be a circle). The basic ideas of the method involve first the proof of existence of an extremal configuration by well-known arguments of equicontinuity and domain convergence. Preliminary differentiability properties of the boundary are then studied by means of level curve variations and symmetrization, while the technique of interior variations, developed by the authors in connection with other problems of analysis, provides the variational condition on the boundary. The final step yields the analyticity of the extremal curves and extremal functions on the boundary. This is accomplished by solving a set of non-linear integral equations whose derivation is based on the theory of the Riemann function and the Cauchy problem for linear partial differential equations in the complex domain. This idea has already been exploited by H. Lewy, and by the authors jointly with H. Lewy [Ann. of Math. (2) 56, 560–602 (1952); these Rev. 14, 810] in their solution of the axially symmetric cavity problem. The paper closes with a short survey of further problems of applied mathematics for which the method is appropriate. *D. Gilbarg* (Bloomington, Ind.).

Browder, Felix E. Assumption of boundary values and the Green's function in the Dirichlet problem for the general linear elliptic equation. Proc. Nat. Acad. Sci. U. S. A. 39, 179–184 (1953).

Browder, Felix E. On the eigenfunctions and eigenvalues of the general linear elliptic differential operator. Proc. Nat. Acad. Sci. U. S. A. 39, 433–439 (1953).

Browder, Felix E. Linear parabolic differential equations of arbitrary order; general boundary-value problems for elliptic equations. Proc. Nat. Acad. Sci. U. S. A. 39, 185–190 (1953).

In two previous notes [same Proc. 38, 230–235, 741–747 (1952); these Rev. 14, 174, 473] the author has established the Fredholm alternative for the solution of the Dirichlet problem of an elliptic equation $Ku=\eta$ of order $2m$ in a domain D in n-dimensional space. The boundary condition imposed required that for a given function g the function $u-g$ and its derivatives of order $<m$ vanish in a certain generalized sense on the boundary of D. The question of assumption of boundary values in the ordinary sense was left open. The present first note contains some extensions of those results. Among the new results is a proof that in the 2-dimensional case the solution of the Dirichlet problem satisfies the boundary conditions in the ordinary sense (under suitable regularity conditions on K, g, η and the boundary of D). In higher dimensions it is proved only that u assumes the boundary data in the ordinary sense if u is assumed to be of class C_{m-1} in the closure of D. In addition, it is shown that for domains D with a boundary of measure 0, for which the solution of the Dirichlet problem is unique, there exists a Green's function for K.

The second note contains a proof of the completeness of the eigenfunctions of a linear elliptic differential operator K of order $2m$, which is not required to be self-adjoint. Here the eigenfunctions are taken with respect to an operator B of lower even order $2s$, which is assumed to be self-adjoint and of "single sign". The latter property is defined by the requirement that $\int_D Bf\cdot f^* dx$ is bounded either above or below by a multiple of the L^2 norm of the derivatives of f of order s, for f with compact support in D. Eigenfunctions of different orders have to be used. Those of the first order are solutions of $K\Phi-\lambda\Phi=0$ with homogeneous boundary conditions (in the generalized sense). Those of order ν are solutions of $K\Phi-\lambda B\Phi=B\psi$ with the same boundary conditions, where ψ is an eigenfunction of order $\nu-1$.

In addition, it is shown that the eigenfunctions of K are complete in $L^p(D)$ for $p\leqq 2n/(n-2m)$ and closed in $L^1(D)$ for $2m>n$. Estimates are also given for the relative magnitudes of the real and imaginary parts of the eigenvalues.

The third note treats the initial boundary-value problem for the parabolic equation $(-1)^{m+1}Ku+\eta=\partial u/\partial t$, where K is a self-adjoint elliptic operator of order $2m$ in the bounded domain D. Solutions with given values of u in $L^2(D)$ for $t=0$ and with homogeneous Dirichlet conditions (in the generalized sense) for $t>0$ on the boundary of D are desired. As a consequence of the completeness of the eigenfunctions of K established in the previous notes, it is proved that (under suitable regularity assumptions) a solution u exists in D, and is of class C_{2m} in the space variables and of class C_∞ in t for $t>0$, provided the Dirichlet problem for K in D has no trivial solution. [A similar result by Milgram and Lax (to appear in Annals of Math. Studies, vol. 33) without the assumption of self-adjointness is mentioned].

A second part of this note deals with generalizations of the Dirichlet problem for elliptic equations to other types

of linear boundary conditions. A suitable generalized sense is given to such boundary conditions which does not involve regularity of u on the boundary. Definitions (which involve norms of various types and are too complicated to be quoted here) are given for a boundary-value problem to be "proper" or "strongly proper". For proper boundary-value problems the first and second Fredholm alternatives are proved. For strongly proper ones the discreteness and finite multiplicity of the eigenvalues are established. *F. John.*

Hellwig, Günter. Bemerkungen zu der Satzgruppe von Hilbert über Systeme elliptischer Differentialgleichungen. Math. Z. 55, 276–283 (1952).

Let F be a domain in the (x, y)-plane bounded by a simple closed curve R with a continuous curvature. Let p, q, k, l be continuously differentiable functions on the closed domain $F+R$, $f(s)$ a twice continuously differentiable function of the arc length s on R. The author considers the problem of finding a pair of functions u, v continuous on $F+R$ and continuously differentiable in F which satisfy the system of two partial differential equations,

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=p(x, y)u+q(x, y)v,$$

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=k(x, y)u+l(x, y)v$$

in F, while on the boundary $u(s)=f(s)$. In his work on integral equations, Hilbert proved that if this problem has only the solution $u\equiv 0$, $v\equiv 0$ for the boundary function $f(s)\equiv 0$, then it has a solution for every boundary function $f(s)$. In case there are such null solutions, then solutions exist for $f(s)$ if $f(s)$ satisfies a finite number of integral conditions. In this paper the author shows that the first possibility never occurs, since there always exists a pair of non-zero functions u and v which solve the problem for $f(s)\equiv 0$.

F. E. Browder (Princeton, N. J.).

Hellwig, Günter. Randwertprobleme nichtlinearer elliptischer Differentialgleichungssysteme erster Ordnung mit Anwendungen auf die Verbiegung von elliptisch gekrümmten Flächenstücken. Math. Nachr. 8, 13–30 (1952).

The boundary-value problem $\mu_1(s)u(s)+\mu_2(s)v(s)=u_0(s)$ on the boundary R of the plane domain F is considered for the general first-order quasi-linear elliptic system in two independent variables. If the domain F is bounded by the smooth Jordan curve R and is sufficiently small, the results of a previous paper by the author on linear elliptic systems [cf. the preceding review] are applied to obtain solutions of the boundary-value problem with perturbed boundary conditions close to a given solution. An application is given to the deformation of a small piece of surface which has elliptic curvature. *F. E. Browder* (Princeton, N. J.).

Hellwig, Günter. Das Randwertproblem eines linearen elliptischen Systems. Math. Z. 56, 388–408 (1952).

Let F be a domain in the plane bounded by a smooth Jordan curve R. The author considers the elliptic system of equations on F,

$$\text{1)}\qquad \begin{aligned} &a^1u_{x^1}+a^2u_{x^2}+b^1v_{x^1}+b^2v_{x^2}-du-ev-f=0\\ &\bar{a}^1u_{x^1}+\bar{a}^2u_{x^2}+\bar{b}^1v_{x^1}+\bar{b}^2v_{x^2}-\bar{d}u-\bar{e}v-\bar{f}=0\end{aligned}$$

with suitable regularity conditions on the coefficients in $F+R$. For $\mu_1(s)$, $\mu_2(s)$ smooth functions of the arc length on R, the boundary-value problem $\mu_1(s)u(s)+\mu_2(s)v(s)=u_0(s)$ is set with an arbitrary smooth function u_0 on R. Let k be a differentiable positive function on $F+R$, γ an arbitrary real number. Suppose that the Kronecker characteristic of the vector field with components μ_1, μ_2 on R is zero and therefore that μ_1, μ_2 can be extended to continuous functions φ_1, φ_2 on $F+R$ for which $\varphi_1^2+\varphi_2^2$ never vanishes. The author shows that for three suitably chosen functions p, q, r on $F+R$ defined from the coefficients of (1), and for the domain F sufficiently small, there exists exactly one pair u, v of solutions of the boundary-value problem and of the system (1) satisfying the normalization condition,

$$\int_F \{(\varphi_1 p-\varphi_2 q)u+(\varphi_1 r-\varphi_2 p)v\}k\,dA=\gamma.$$

With further uniqueness conditions, for a special class of systems of the form (1) and for a special class of boundary problems, some results in the large are established.

F. E. Browder (Princeton, N. J.).

Haack, Wolfgang. Allgemeine Randwertprobleme für Differentialgleichungen vom elliptischen Typus. Die Überführung des Randwertproblems für Systeme elliptischer Differentialgleichungen auf Fredholmsche Integralgleichungen. II. Math. Nachr. 7, 1–30 (1952).

Let

$$\text{(1)}\quad a^iu_i+b^iv_i+cu+dv+e=0,\quad \bar{a}^iu_i+\bar{b}^iv_i+\bar{c}u+\bar{d}v+\bar{e}=0$$

be a linear elliptic system of differential equations for functions u, v of (x_1, x_2) with derivatives u_i, v_i in a simply connected domain F. [The summation convention with $i=1, 2$ is employed.] For what follows F is assumed to be sufficiently small, and to have a boundary R, whose curvature varies continuously with the arc length s. The boundary conditions considered are of the form

$$\text{(2)}\qquad a(s)u+b(s)v=f(s)\quad\text{on}\quad R.$$

Boundary conditions are classified according to the index n of the vector field (a, b) on R (i.e., n = number of complete turns described by the vector along R). The present paper is concerned with the cases $n=0, 1$, which are referred to as "first" and "second" boundary-value problems. It is proved that the solution of the second boundary value problem is unique, but exists only if f satisfies an integral condition. In contrast to that, the first boundary value problem possesses a one-parameter family of solutions for all f; in particular, the homogeneous problem ($e=\bar{e}=f=0$) has a solution with $u^2+v^2\neq 0$ in $F+R$. [See the paper of Nitsche reviewed second below.]

In the present paper less use is made of the invariant notation developed by Haack and Hellwig [Math. Nachr. 4, 408–418 (1951); these Rev. 12, 830] and of integral equations. Instead (1) is reduced to the normal form

$$\text{(3)}\qquad u_x-v_y=au+bv+c,\quad u_y+v_x=\bar{a}u+\bar{b}v+\bar{c}$$

by means of a solution of Beltrami's equation. Then use is made of the existence and uniqueness of the solution of the Dirichlet problem for a single second-order linear or quasi-linear equation [see Courant and Hilbert, Methoden der mathematischen Physik, Bd. 2, Springer, Berlin, 1937, Chap. IV]. This accounts for restrictions on the size of F. Moreover, the general boundary condition (2) is reduced by a suitable linear transformation on u, v to the special case where either u itself or the tangential component of (u, v) is given on R. Use of a suitable solution of the adjoint system to (1) or (3) permits reduction to the Dirichlet problem for a linear second-order equation. As an illustration the author

shows how the second boundary-value problem for the Cauchy-Riemann equations can be solved by quadratures from a knowledge of Green's function for F of the Dirichlet problem for the Laplace equation.

The linear second order equation

$$\phi_{xx}+\phi_{yy}+A\phi_x+B\phi_y+E\phi+C=0 \tag{4}$$

can be reduced to a system (3). This enables us to solve the "oblique derivative" problem, $p^2\phi_x+p^1\phi_n=f(s)$ on R, for (p^1, p^2) of indices 0, -1, in case the coefficient E in (4) vanishes. For $E\neq 0$ less complete results are obtained, and only boundary problems of the form $f(s)=\alpha\phi_y+\beta\phi_x-P\phi$ with special functions P can be solved by this method.

F. John (New York, N. Y.).

Haack, Wolfgang. Randwertprobleme höherer Charakteristik für ein System von zwei elliptischen Differentialgleichungen. Math. Nachr. 8, 123–132 (1952).

This paper deals with a system in the standard form of formula (3) of the preceding review with a boundary condition (2) where now, however, the index of the vector field (a, b) with respect to the boundary curve R can be an arbitrary integer n. It is shown that for sufficiently small domains F the boundary-value problem for negative n has a $(2|n|+1)$-parameter family of solutions. $2n$ of the parameters can be fixed by prescribing n common zeros of the functions u, v, provided the coefficients a, b in (2) satisfy certain involved inequalities. For positive n the solution is unique, but exists only if f, c, $\bar{c}$ satisfy certain integral conditions or if the solution is permitted to have "poles".

The proof is based on results of Hellwig [see the paper reviewed second above] on solutions of second order elliptic equations by the method of integral equations, and on the fact that boundary-value conditions of the type (2) for vector fields (a, b) with the same index can be transformed into each other.

F. John (New York, N. Y.).

Nitsche, Joachim. Das erste Randwertproblem eines linearen elliptischen Differentialgleichungssystems. Math. Nachr. 7, 31–33 (1952).

[See the two preceding reviews.] Proof that the first boundary-value problem for equations (3) in a sufficiently small domain possesses a one-parameter family of solutions. The proof is accomplished by reduction to the Dirichlet problem for a non-linear second order equation. (Haack in the paper reviewed second above makes use of this argument of Nitsche.)

F. John (New York, N. Y.).

Nitsche, Joachim. Beiträge zum Randwertproblem quasilinearer elliptischer Differentialgleichungssysteme. Math. Nachr. 7, 35–54 (1952).

The author derives a "normal form" for a quasi-linear elliptic system of 2 first order equations for 2 unknown functions u, v, of x^1, x^2, analogous to the one given by Haack and Hellwig [Math. Nachr. 4, 408–418 (1951); these Rev. 12, 830] for linear systems. Solving an auxiliary linear system of elliptic equations in the independent variables u, v, one can reduce the original quasi-linear system to the form

$$\begin{aligned} \nabla_1 U-\nabla_2 V&=F(U, V, x^1, x^2),\\ \nabla_2 U+\nabla_1 V&=G(U, V, x^1, x^2) \end{aligned} \tag{1}$$

for the new dependent variables U, V. Here ∇_1, ∇_2 are invariant derivatives with respect to two Pfaffian forms $\omega^1=m_i dx^i$, $\omega^2=n_i dx^i$ in the Riemann space with metric $ds^2=(\omega^1)^2+(\omega^2)^2$, and the m_i, n_i are functions of U, V, x^1, x^2. Equations (1) lead to integral identities involving arbitrary functions of U, V, x^1, x^2 and also derivatives of U, V. Choosing these arbitrary functions in their dependence on U, V as solutions of a suitable linear elliptic system, one obtains integral identities for U, V, which are free of derivatives of U, V. A formal scheme is indicated by which boundary value problems for (1) may be solved by reduction to non-linear integral equations.

The reduction to normal form is carried out for three examples: the equation of minimal surfaces, the Gauss-Codazzi equations for surfaces of positive curvature, and the differential equations of plane irrotational compressible flows.

F. John (New York, N. Y.).

Bochner, S. Zeta functions and Green's functions for linear partial differential operators of elliptic type with constant coefficients. Ann. of Math. (2) 57, 32–56 (1953).

Let T be a polynomial in k variables with complex coefficients of elliptic type, i.e., $T=T_{2h}+R$, where T_{2h} is of degree $2h$, R of lower degree, $T_{2h}(t)>0$ for $t\neq 0$, $T(t)\neq 0$ for $t\neq 0$. Let Q be a homogeneous polynomial. The author considers the functional properties of

$$\zeta(s, x)=\lim_{\epsilon\to 0}\sum{}' e^{2\pi i(x,n)-\epsilon n^2}Q(n)/T(n)^s,$$

the sum being taken over integer points in E^k with the exception of the origin. It is shown that, for any complex s, $\zeta(s, x)$ is defined as a limit for $x\neq 0$ uniformly in the neighborhood of each point and lies in C^∞, while for $x\neq 0$, ζ is an entire function in s. For $x=0$, the single pole of ζ in the s-plane is studied. In the special case in which $Q=1$ and T is homogeneous, $\zeta(s, x)$ is shown to be analytic in x for $x\neq 0$ for every complex s. For $s=1$ in the latter case, it is described as the Green's function of the elliptic differential operator with constant coefficients $T(\partial/\partial x)$ on the multitorus T^k.

F. E. Browder (Princeton, N. J.).

Pucci, Carlo. Maggiorazione della soluzione di un problema al contorno, di tipo misto, relativo a una equazione a derivate parziali, lineare, del secondo ordine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 13, 360–366 (1952).

Let A be a region in r-dimensional euclidean space and suppose the boundary of A is decomposed into two parts $H+I=\mathfrak{F}A$. It is assumed that for each point P of $\mathfrak{F}A$ there is a line segment lying entirely in A except for P. Suppose $u(x_1, x_2, \cdots, x_r)$ is continuous in $A+\mathfrak{F}A$, has continuous second derivatives in $A+I$, and satisfies the elliptic-parabolic equation

$$\sum_{h,k=1}^{r} a_{hk}(X)\frac{\partial^2 u}{\partial x_h\partial x_k}+\sum_{h=1}^{r} b_h(X)\frac{\partial u}{\partial x_h}+c(X)u=f(X)$$

in A subject to the conditions

$$\alpha(X)\frac{du}{dl}+\beta(X)u=\gamma(X),\quad X\in H \quad\text{and}\quad \frac{du}{dl}=0,\quad X\in I.$$

It is further assumed that $\beta(X)\neq 0$, $X\in H$; $c(X)<0$, $X\in A+I$; $\alpha(X)\beta(X)\leqq 0$, $X\in H$. Then the following inequality holds in $A+\mathfrak{F}A$:

$$\min\left\{\min_{A+I}\left[\frac{f}{c}\right], \min_H\left[\frac{\gamma}{\beta}\right]\right\}\leqq u(X)$$

$$\leqq \max\left\{\max_{A+I}\left[\frac{f}{c}\right], \max\left[\frac{\gamma}{\beta}\right]\right\}.$$

This generalizes previous results of the author [same Rend. (8) 11, 334–339 (1951); these Rev. 13, 946].
M. H. Protter (Berkeley, Calif.).

Pogorzelski, W. Remarques sur un problème mixte concernant l'équation des télégraphistes. Prace Mat.-Fiz. 48, 59–66 (1952).

The mixed initial-boundary-value problem for the telegrapher's equation is discussed. The results are well known.
M. H. Protter (Berkeley, Calif.).

Difference Equations, Special Functional Equations

Sobolev, S. L. On the uniqueness of solution of difference equations of elliptic type. Doklady Akad. Nauk SSSR (N.S.) 87, 179–182 (1952). (Russian)

Sobolev, S. L. On a difference equation. Doklady Akad. Nauk SSSR (N.S.) 87, 341–343 (1952). (Russian)

In the first note the author considers the difference equation

$$Lu_{m,n}=\tfrac{1}{4}\{u_{m+1,n+1}+u_{m-1,n-1}+u_{m+1,n-1}+u_{m-1,n+1}-4u_{m,n}\}=0,\quad -\infty<m<+\infty,\ -\infty<n<+\infty,$$

and proves that if a solution of this equation grows more slowly than $(m^2+n^2)^{1/2}$ at infinity, then this solution must be a constant. In the second note the author gives an explicit formula for a solution $w_{m,n}$ of the difference equation $(-\infty<m<+\infty,\ -\infty<n<+\infty)$

$$\tfrac{1}{4}\{w_{m+1,n+1}+w_{m-1,n-1}+w_{m+1,n-1}+w_{m-1,n+1}-4w_{m,n}\}=\begin{cases}1, & m^2+n^2=0,\\ 0, & m^2+n^2>0,\end{cases}$$

which satisfies $w_{0,0}=0$ and is of the order of $\ln (m^2+n^2)^{1/2}$ at infinity. (That there is at most one such function follows from the preceding theorem.) The formula is

$$w_{m,n}=-\frac{1}{\pi^2}\oint_{|u|=1}\left(\oint_{|v|=1}\frac{u^mv^n-1}{u^2v^2+u^2+v^2+1-4uv}dv\right)du.$$

It is shown that for large $(m^2+n^2)^{1/2}$ there is a constant A such that

$$\left|w_{m,n}-\frac{2}{\pi}(C+\ln 2+\ln (m^2+n^2)^{1/2})\right|<\frac{A}{(m^2+n^2)^{1/2}},$$

where C is Euler's constant. [In a letter to the editor, same journal, 88, 740 (1953), the author states that equivalent results are contained in A. Stöhr, Math. Nachr. 3, 208–242, 295–315, 330–357 (1950); these Rev. 12, 711.]
J. B. Diaz (College Park, Md.).

Jarden, Dov. Recurring sequences of order 3. Riveon Lematematika 6, 41–44 (1953). (Hebrew)

The author discusses difference equations of the type

$$w_n=aw_{n-3}+bw_{n-2}+cw_{n-1}$$

and obtains a number of algebraic recursion relations. He also obtains some recursive congruence relations in case a, b, c, w_0, w_1, w_2 are rational.
E. G. Straus.

Boas, R. P., Jr. Functions which are odd about several points. Nieuw Arch. Wiskunde (3) 1, 27–32 (1953).

The main results are as follows. If the real-valued and periodic function $f(t)=f(t+1)$ satisfies the functional equation $f(x+t)+f(x-t)=2f(x)$ of Jensen for almost all x and (1_1) for two values of x whose difference is irrational, resp. (1_2) for an infinite number of x-values in (0, 1), resp. (2) for two rationally independent x's, resp. (3) for a set of x's having a positive measure, then (1_1) or (1_2) and measurability implies that $f(t)$ is almost everywhere constant, (2) and boundedness on a set of positive measure imply that $f(t)$ is bounded (but not necessarily constant) almost everywhere, and, finally, (3) and boundedness on a set of positive measure imply that $f(t)$ is constant almost everywhere. It is also proved that if $f(t)$ is periodic, integrable, and satisfies $f(x+t)+f(x-t)-2f(x)=h_x(t)$ for an x-set of positive measure with $h_x(t)$ analytic in the strip $|I(t)|\leqq\delta>0$, then $f(t)$ is analytic.
J. Aczél (Debrecen).

Ewing, G. M., and Utz, W. R. Continuous solutions of the functional equation $f^n(x)=f(x)$. Canadian J. Math. 5, 101–103 (1953).

Two classes I, II of functions are defined as follows. I: (a) $f(x)$ is continuous for all real x; (b) $f(x)\equiv x$ on a connected subset S of the x-axis; (c) if $g=\inf f(x)$, $G=\sup f(x)$, both taken over S, then also $g\leqq f(x)\leqq G$ for all x. II: (a) as in I, and either (b) $f[f(x)]\equiv x$ or (c) $f[f(x)]\equiv x$ on a non-degenerate closed interval $[a, b]$, with $f(a)=b$, $f(b)=a$, and $a\leqq f(x)\leqq b$. It is then shown that the continuous real solutions of the iterate functional equation $f^{(n)}(x)=f(x)$, $n\geqq 2$, are the functions of class I if n is even and of classes I and II if n is odd. It is noted by example that equation $f^{(n)}(x)=f^{(m)}(x)$ may have solutions not in I or II.
I. M. Sheffer (State College, Pa.).

Integral Equations

Hampel, R. Quelques remarques se rapportant aux noyaux itérés dans l'espace à p-dimensions. Prace Mat.-Fiz. 48, 111–128 (1952).

The kernels considered are of the form

$$K(A, B)=H(A, B)/r^\alpha,$$

where A and B are points in a p-dimensional rectangle, r is the distance between A and B, $\alpha<p$ and $0<H(A, B)<W$. Bounds for the iterates of such kernels are obtained based on the inequality between the arithmetic and geometric mean, i.e., $r^\alpha\geqq p^\alpha\prod_{i=1}^{n}(x_i-t_i)^{\alpha/p}$ where $A=(x_1, \cdots, x_n)$ and $B=(t_1, \cdots, t_n)$. The value $\alpha=np/(n+1)$ plays a role in the bounds for the nth iterated kernel. There is a discussion of the diophantine equation $\alpha=np/(n+1)$, α being an integer, $n\leqq N$, and $\alpha+1\leqq p<P$, P and N being given, finite or infinite.
T. H. Hildebrandt (Ann Arbor, Mich.).

Germay, R. H. Extension à des équations intégro-différentielles récurrentes à plusieurs variables indépendantes, du théorème de Cauchy relatif à l'existence des intégrales des équations aux dérivées partielles du premier ordre. Bull. Soc. Roy. Sci. Liège 21, 491–496 (1952).

Germay, R. H. Extension à des systèmes d'équations intégro-différentielles récurrentes à plusieurs variables indépendantes du théorème de Cauchy relatif à l'existence des intégrales des systèmes d'équations aux dérivées partielles du premier ordre. Bull. Soc. Roy. Sci. Liège 22, 2–10 (1953).

Both these notes extend previous results on existence theorems to "recurrent" integro-differential equations. The

first paper has reference to a note in the same Bull. 2, 118–121, 157–160 (1933); the second paper to a note in the same Bull. 2, 190–194, 212–217 (1933). *I. A. Barnett.*

Germay, R. H. Sur les solutions infiniment voisines des équations intégro-différentielles récurrents de forme normale. Bull. Soc. Roy. Sci. Liège 22, 64–76 (1953).

This note extends previous results on "recurrent" integro-differential equations to similar equations containing parameters [these Rev. 12, 500; 13, 345]. *I. A. Barnett.*

Wolska, J. Sur les équations intégrales et intégro-différentielles à singularité polaire. Prace Mat.-Fiz. 48, 27–44 (1952).

As a necessary preliminary to the resolution of the equations considered by the author, a study is made of a transformation of Poincaré [Leçons de mécanique céleste, t. III, Gauthier-Villars, Paris, 1910], relating to integrals of the form $\int_0^a N_1(x, y)[\int_0^a N(y, z)\phi(z)dz]dy$, where N, N_1 are analytic in each of the variables and have, as their only singularities in an infinite band Ω containing the axis of reals, poles of first order $x=y$; ϕ is analytic in a domain containing the real segment $[0, a]$. This study involves Cauchy principal values. $N(x, y)$ is termed a closed, singular kernel if the equation $\int_0^a N(x, y)h(y)dy=0$ $(0<x<a)$ has no analytic solutions besides the trivial one $h=0$. A necessary and sufficient condition is found in order that N be closed. The integro-differential equation solved is of the form $\int_0^a \sum_{\nu=0}^{n}{}' N_\nu(x, y)\phi^{(\nu)}(y)dy=f(x)$; here the N_ν, of period a, are of the type of N; f is analytic in Ω and is of period a; ϕ is the unknown; the integrations are in the sense of principal values; ϕ is subject to the condition: $\phi^{(j)}(0)=C_j$ $(j=0, \cdots, n)$, where the C_j are assigned. The problem is reduced to that of an equation "mixed" in the sense that one of the integrals has fixed limits and another one has variable limits; this finally gives a regular Fredholm equation. A similar study is made of a nonlinear system in the sense of principal values; the system is such as to be susceptible to the method of successive approximations, yielding a unique solution. *W. J. Trjitzinsky* (Urbana, Ill.).

Pogorzelski, W. Le noyau singulier fermé. Prace Mat.-Fiz. 48, 105–110 (1952).

Let $N(x, y)$ be uniform, analytic in a band Ω containing the axis of reals, have a real period a, and have as its only singularity (disregarding multiples of a) in Ω a pole $x=y$. The author terms $N(x, y)$ a closed, singular kernel if the equation $\int_0^a N(x, y)\phi(y)dy=0$ (integrations in the sense of principal values) has no analytic solutions, except the trivial one $\phi=0$. The kernel $\operatorname{cotg} \pi(x-y)/a$ is not closed, while $\operatorname{cotg} \pi(x-y)/a+\mu$ $(\mu\neq0)$ is. Considered are kernels of the form

$$(1)\qquad N(x, y)=A_0(y)\operatorname{cotg}\frac{\pi}{a}(x-y)+\sum_{\nu=1}^{n}A_\nu(x)B_\nu(y),$$

where the A_ν, B_ν are of real period a and are analytic in Ω. Let

$$C_\nu(x)=\int_0^a A_\nu(y)\left[1+\operatorname{cotg}\frac{\pi}{a}(x-y)\right]dy.$$

It is shown that the inequality

$$(2)\qquad \det\left|\int_0^a \frac{B_\nu(z)}{a^2A_0(z)}C_\alpha(z)dz-\delta_{\alpha\nu}\right|\neq0\quad(\alpha, \nu=0, 1, \cdots, n),$$

$\delta_{\alpha\nu}$ being the Kronecker symbol, is sufficient for the closure of the kernel (1); furthermore, the condition (2) is also necessary if the $A_1, \cdots, A_n$ satisfy no relation $\sum_{\nu=1}^{n}p_\nu A_\nu(x)=q$, where the p_ν and q are constants, not all simultaneously zero. A vital role in the above is played by Poincaré's [Leçons de mécanique céleste, t. III, Gauthier-Villars, Paris, 1910] formula for iterated integrals.

W. J. Trjitzinsky (Urbana, Ill.).

Nickel, Karl. Lösung von zwei verwandten Integralgleichungssystemen. Math. Z. 58, 49–62 (1953).

The author solves the two systems of integral equations

$$(1)\qquad \frac{k}{\pi}\sum_{1}^{n}\int_{a_\nu}^{b_\nu} f_\nu(y_\nu)\operatorname{ctg} k(x_j-y_\nu)dy_\nu=g_j(x_j),$$

$$(j=1, \cdots, n)$$

$$(2)\qquad \frac{k}{\pi}\sum_{1}^{n}\int_{a_\nu}^{b_\nu}\frac{f_\nu(y_\nu)}{\sin k(x_j-y_\nu)}dy_\nu=g_j(x_j)$$

by reducing them to the case $n=1$, which is the case resolved previously [Nickel, Z. Angew. Math. Mech. 31, 285–286 (1951); Ing.-Arch. 20, 6–7 (1952); these Rev. 14, 54; J. Dörr, Ing.-Arch. 19, 66–68 (1951); these Rev. 13, 44]. The intervals (a_ν, b_ν) $(\nu=1, \cdots, n)$ are so labeled that $a_\nu<b_j$ for $\nu\leq j$, $a_\nu>b_j$ for $\nu>j$; the Lebesgue integrals

$$\int_{a_\nu}^{b_\nu}|g_\nu(x)|^p[(b_\nu-x_\nu)(x_\nu-a_\nu)]^{p/2}dx,$$

are assumed to exist, real $p>1$ being chosen arbitrarily; the parameter k is real or purely imaginary; $|k|\leq\pi/(b_n-a_1)$. The systems (1) and (2) are reduced to

$$k\pi^{-1}\int_0^b f(y)\operatorname{ctg} k(x-y)dy=g(x);$$

for $k=1/2$ this is an equation solved by Hilbert [Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, Teubner, Leipzig-Berlin, 1912]. The method is that of complete induction. Reduction of (1) from n to $n-1$ equations gives an equation of the form (2), and conversely. The most general solutions of (1) and (2), respectively, are given explicitly. *W. J. Trjitzinsky* (Urbana, Ill.).

San Juan, Ricardo. Résolution d'un système infini d'équations linéaires. C. R. Acad. Sci. Paris 236, 1841–1843 (1953).

It is shown that if in the system of equations

$$(1)\qquad \sum_{\nu=1}^{\infty}a_\nu m_\nu^n=C_n\quad(n=0, 1, \cdots)$$

the conditions $\limsup |C_n|^{1/n}<\infty$, $m_{\nu+1}/m_\nu\geq(1+1/\nu)^\tau$, $\tau>2$, hold, then (1) has a solution $\{a_\nu\}$ for which (2) $a_\nu=o(e^{-k\nu})$, with $0<k<\pi\cot(\pi/\tau)$. This generalizes a result of de la Vallée Poussin [Rice Inst. Pamphlet 12, 101–172 (1925), pp. 165–170]. The method (like that of de la Vallée Poussin) considers first the reduced system $\sum_{\nu=1}^{r}a_\nu m_\nu^n=C_n$ $(n=0, 1, \cdots, r-1)$, whose solution can be expressed as a contour integral in the complex plane, and then passes to the limit $(r\to\infty)$. It is also shown that if $\tau<2$, there is no solution satisfying (2) with $k>0$. *I. M. Sheffer.*

Functional Analysis

Mann, W. Robert. Mean value methods in iteration. Proc. Amer. Math. Soc. 4, 506–510 (1953).

Let E be a convex compact subset of a Banach space, and T a map of E into itself. The author considers the follow-

ing iteration procedure: let a_{ij} $(i, j=1, 2, \cdots)$ be given non-negative constants subject to the restrictions $a_{11}=1$, $a_{ij}=0$ for $j>i$, and $\sum_{j=1}^{i} a_{ij}=1$. [The author communicates that the following condition should also be included: $\lim_{i\to\infty} a_{ij}=0$ for all j.] With x_1 being an arbitrarily given element of E, two sequences x_n, v_n of elements of E are defined by the recursion $x_{n+1}=T(v_n)$, $v_n=\sum_{k=1}^{n} a_{nk}$. (Obviously, this procedure reduces to the usual iteration $x_{n+1}=T(x_n)$ if the matrix a_{ik} is the identity matrix.) The following theorems are proved. 1) If either of the two sequences converges, so does the other; their limit is then the same and a fixed point of T. 2) Let X and V denote the set of limit points of the sequences x_n, v_n respectively, and let the a_{ij}, in addition to the assumptions made above, satisfy

$$\lim_{n\to\infty} a_{nn}=\lim_{n\to\infty} \sum_{k=1}^{n} |a_{n+1,k}-a_{nk}|=0.$$

Then X and V are closed connected sets. 3) The closed convex hull of X contains V. *E. H. Rothe.*

Temple, G. Theories and applications of generalized functions. J. London Math. Soc. **28**, 134–148 (1953).

This is a primarily expository paper devoted to a presentation of the Schwartz theory of distributions from a weak function point of view. Particular reference is made to Mikusiński's approach to generalized functions by means of weak convergence [Studia Math. **11**, 41–70 (1949); these Rev. **12**, 189]. *I. E. Segal* (Chicago, Ill.).

Cameron, R. H., and Hatfield, C. On the summability of certain series for unbounded nonlinear functionals. Proc. Amer. Math. Soc. **4**, 375–387 (1953).

In an earlier paper [Bull. Amer. Math. Soc. **55**, 130–145 (1949); these Rev. **10**, 462] the authors have obtained a result on infinite-dimensional Abel summability of the orthogonal development of a bounded nonlinear functional $F(x)$ in terms of a set of closed orthonormal functionals, the Fourier-Hermite functionals. In the present paper the authors remove the condition of boundedness, and replace it by the condition that $F(x)$ be a functional on $L_2(C)$ such that $|F(x)|<B \exp \{A\int_0^1 x^2(t)dt\}$. The proof, while similar in some respects to that of the earlier paper, needs a different and more refined argument because of the presence of the exponential factor in the bound of F. *W. T. Martin.*

Block, I. Edward. Kernel functions and class L^2. Proc. Amer. Math. Soc. **4**, 110–117 (1953).

Let B be a domain in the complex z-plane and let $L^2(B)$ be the Hilbert space of functions $f(z)$ of summable square on B. If the inner product (f, g) of two functions of $L^2(B)$ is defined by $\int\int_B fg^* dxdy$ $(z=x+iy)$ and $K=K(z, \zeta)$ denotes the Bergman kernel function of B, then the analytic function $f_1(\zeta)=(f, K)$ is the projection of $f(z)$ on the subspace $\bar{L}^2(B)$ of $L^2(B)$ which is composed of analytic functions. The function $f_2(z)$ defined by $f(z)=f_1(z)+f_2(z)$ belongs to the orthogonal complement $\hat{L}^2(z)$ of $\bar{L}^2(B)$.

With a view to setting up an operator which will determine the projection $f_2(z)$ of $f(z)$ on $\hat{L}^2(B)$, the author studies the kernel $L=L(z, \zeta)$, introduced by Schiffer [Duke Math. J. **13**, 529–540 (1946); these Rev. **8**, 371], which is connected with $K(z, \zeta)$ by the relation $L(z, \zeta)dz=-[K(z, \zeta)dz]^*$ on the boundary of B. If $L(f)=(f, L)$ (the improper integral defined as a Cauchy principal value), he then shows that $L[L(f)]=f_2(z)$ if $f(z)$ is defined as zero outside B. He also shows that the subspace of functions which satisfy on B a Lipschitz condition of order α $(0<\alpha<1)$ is invariant under the L-transformation. *Z. Nehari* (St. Louis, Mo.).

Klee, Victor L., Jr. Convex bodies and periodic homeomorphisms in Hilbert space. Trans. Amer. Math. Soc. **74**, 10–43 (1953).

The author studies certain topological properties of normed linear spaces. Hilbert space receives special attention. The results obtained include the following: If E is a nonreflexive space or an infinite-dimensional l^p space, then there is a homeomorphism of period two without fixed points of E onto itself which takes the unit cell (set of $x \in E$ for which $\|x\|\leq 1$) onto itself. Every hyperplane in E is homeomorphic with the unit sphere. If X is an arbitrary compact subset of E, then there is an isotopy whose initial transformation is the identity map on E and whose terminal transformation is a homeomorphism of E onto $E-X$. For Hilbert space $\mathfrak{H}$, it is shown that $\mathfrak{H}$ is homeomorphic with its unit cell and with its unit sphere. If B is a closed convex body in $\mathfrak{H}$, then B is homeomorphic with $\mathfrak{H}$. If X is an arbitrary finite polytope in $\mathfrak{H}$ and $n\geq 2$, then $\mathfrak{H}$ admits a homeomorphism of period n with X as its set of fixed points. One-third of the paper (chapter I) is devoted to the proof of a theorem on the existence of certain isotopies in nonreflexive spaces. The previously mentioned results are derived with the help of this isotopy theorem. Chapter IV is devoted to proofs of further results and to the statement of some unsolved topological problems concerning linear spaces. There is an appendix dealing with questions of convexity. A convex body is called strictly convex if its boundary contains no line segments and smooth if it has a unique supporting hyperplane at each boundary point. If S denotes one of these adjectives, let S^* denote the other. Many results are obtained concerning the relation of S to S^*. For example, a reflexive space E is S if and only if E^* is S^*. Also, every separable reflexive E is both S-able and S^*-able. A space E is defined to be S-able if E is isomorphic to an S-space. *E. R. Lorch* (Rome).

Tagamlitzki, Y. Übertragung des Minkowskischen Stützebenensatzes auf Hilbertsche Räume. C. R. Acad. Bulgare Sci. **4** (1951), no. 2–3, 5–8 (1953). (Russian. German summary)

Let H be a not necessarily separable real Hilbert space. Let K and L be subsets of H which are strongly closed and have the property that a, $b \in K$ (L) and α, $\beta\geq 0$ imply $\alpha a+\beta b \in K$ (L). Here K and L are said to be convex closed cones. Suppose further that K is pointed, i.e., that $(k_1, k_2)>0$ for all k_1, $k_2 \in K$ which are distinct from 0. Then, if $K\cap L=\{0\}$, there exists an element $s \in H$ such that $(s, k)\geq 0$ for all $k \in K$ and $(s, l)<0$ for all $l \in L$ which are different from 0. This generalizes a theorem of Minkowski concerning planes of support for convex sets in finite-dimensional spaces [Ges. Abh., Bd. 2, Teubner, Leipzig-Berlin, 1911, pp. 137–229]. It is also to be compared with the weaker separation theorems known for general locally convex topological linear spaces [e.g., N. Bourbaki, Eléments de mathématique, XV, Première partie, Livre V, Chap. II, §3, Actualités Sci. Ind., no. 1189, Hermann, Paris, 1953; these Rev. **14**, 880]. *E. Hewitt* (Seattle, Wash.).

Inzinger, Rudolf. Faltungsgeometrie im Hilbertschen Raume und in der Menge der stützbaren Bereiche einer Ebene. Monatsh. Math. **56**, 105–125 (1952).

The author has on previous occasions [Monatsh. Math. **53**, 227–250, 302–323 (1949); Univ. Roma. Ist. Naz. Alta

Mat. Rend. Mat. e Appl. (5) **10**, 140–155 (1951); these Rev. **11**, 455; **12**, 46; **14**, 184] studied the relations between the geometry of abstract Hilbert space and the totality of support functions $a(\phi)$ in a plane. The relation arises in the following way: If A is a convex curve in the plane, it has a support function $a(\phi)$, $0 \leq \phi \leq 2\pi$. The general support function is obtained by taking linear combinations of these (allowing negative coefficients). These are dense in a Hilbert space $\mathfrak{H}$. Thus a function $a(\phi)$ represents both an element of $\mathfrak{H}$ and a domain with support function (stützbarer Bereich). If $t(\phi)$ is a fixed vector and $x(\phi)$ is variable, the convolution $y(\phi)=\pi^{-1}\int_0^{2\pi} t(\phi-\lambda)x(\lambda)d\lambda$ is a bounded linear transformation in $\mathfrak{H}$ which may now be considered to be a ring as well as a space. The author extends his previous studies to the geometric interrelations under convolution of $\mathfrak{H}$ and the set of domains with a support function.

E. R. Lorch (Rome).

Kato, Tosio. On the perturbation theory of closed linear operators. J. Math. Soc. Japan **4**, 323–337 (1952).

The paper is another contribution to the study of perturbation of operators in Banach space begun by B.v. Sz. Nagy [Acta Sci. Math. Szeged **14**, 125–137 (1951); these Rev. **13**, 849] and the reviewer [Math. Ann. **124**, 317–333 (1952); these Rev. **14**, 288]. If $P(\epsilon)=\sum P_k\epsilon^k$ is an idempotent depending analytically on a complex parameter ϵ, then, by means of a differential equation for operators, the author constructs an $U(\epsilon)$ such that both $U(\epsilon)$ and $U^{-1}(\epsilon)$ are analytic in ϵ and $P(\epsilon)=U(\epsilon)P(0)U^{-1}(\epsilon)$. If $P(\epsilon)$ happens to be self-adjoint in a Hilbert space for real ϵ, then $U(\epsilon)$ for real ϵ is unitary. He deduces yet another proof for Rellich's principal theorem on the analyticity of eigenvalues in the self-adjoint case. He assumes the point of view of the general theory of analytic functions and proves some very general theorems on the analytic character of perturbed eigenvalues. Estimates of convergence radii are given. *František Wolf.*

Kato, Tosio. Perturbation theory of semi-bounded operators. Math. Ann. **125**, 435–447 (1953).

In this paper the author turns the mathematical investigation of perturbation in a fundamentally different direction. Instead of discussing convergent perturbation power series for eigenvalues and eigenvectors, he turns his attention to asymptotic perturbation series. It promises to establish results of fundamental importance for physicists who operate with a finite number of terms. His results in this paper are obtained for $H_\epsilon=H+\epsilon V$, where H and V are semibounded, and the spectrum $\sigma(H)$, below a certain point is of discrete character with finite-dimensional eigenvalues. He obtains asymptotic expansions with $\epsilon>0$, ending with an $o(\epsilon^2)$ term. There is no loss of generality in supposing H strictly positive definite. Then H^{-1} and H_ϵ^{-1} are bounded operators for small ϵ and so the author succeeds in reducing his problem to one of bounded perturbation. In view of the unboundedness of H and V, the proof is naturally complicated by consideration of domains and ranges of operators that occur. Asymptotic expansions have been considered before in a discussion of perturbations of differential equations by Titchmarsh [Proc. Roy. Soc. London. Ser. A. **200**, 34–46 (1949); these Rev. **11**, 596] and, in an abstract form, the reviewer [Math. Ann. **124**, 317–333 (1952); these Rev. **14**, 288] pointed out the possibility of extending some of his results to asymptotic expansions. Examples show the applicability of the author's methods to singular perturbations known from the theory of differential equations.

František Wolf (Berkeley, Calif.).

Harazov, D. F. On a class of linear equations in Hilbert spaces. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 65–72 (1952). (Russian)

Let T_λ be a linear operator in Hilbert space, defined for λ in a domain Λ of the complex plane, and analytic in Λ in the sense that, for each $\lambda_0 \in \Lambda$,

$$T_\lambda=\sum_{m=0}^{\infty}(\lambda-\lambda_0)^m T_m$$

for all λ in the largest open circle of centre λ_0 and contained in Λ, the series $\sum_{m=0}^{\infty}(\lambda-\lambda_0)^m\|T_m\|$ being convergent for all such λ. Suppose also that each T_m is completely continuous. The author then shows that either $(I-T_\lambda)^{-1}$ exists, where I is the identity operator, for no $\lambda \in \Lambda$ or it exists for all $\lambda \in \Lambda$ except perhaps on a set of points having no limit point in Λ; in particular, the second case occurs if $T_{\lambda_0}=0$ for some $\lambda_0 \in \Lambda$. This result contains as a special case a theorem of Tamarkin [Ann. of Math. (2) **28**, 127–152 (1927)], who discussed integral operators connected with Fredholm integral equations. The author also remarks, without giving details of the proof, that the result still holds if T_λ has a pole at each point of a set M having no limit point in Λ, the principal part of T_λ at each pole being of finite rank.

F. Smithies (Cambridge, England).

Maurin, K. On Parseval equation for almost periodic vectors. Studia Math. **13**, 83–86 (1953).

This note presents a brief, elegant, and elementary proof of the Parseval relation in H. Weyl's theory of almost periodic invariant vectors [Amer. J. Math. **71**, 178–205 (1949); these Rev. **10**, 461, 856]. Let $\mathfrak{H}$ be a (possibly incomplete) Hilbert space with elements $f, g, \cdots$ and inner product (f, g). Let $\mathfrak{H}$ be provided with a second norm $|f|$ such that $\|f\|\leq|f|$. Let Σ be a group of linear transformations of H into itself such that $(\sigma f, \sigma g)=(f, g)$ and $|\sigma f|=|f|$ for all $\sigma \in \Sigma$. If, for a particular $f \in \mathfrak{H}$, the group Σ is compact in the metric topology

$$\rho_f(\sigma, \tau)=|\sigma f-\tau f|,$$

then f is said to be almost periodic. Let f be a fixed almost periodic vector in $\mathfrak{H}$. Then Weyl's fundamental theorem asserts that there exists a finite or countably infinite sequence of pairwise orthogonal subspaces $\mathfrak{H}_\nu$ of $\mathfrak{H}$ each of finite dimension, where $\mathfrak{H}_\nu$ is spanned by the orthonormal vectors $g_{n_\nu+1}, \cdots, g_{n_{\nu+1}}$, and that $f=\sum_\lambda(f, g_\lambda)g_\lambda$. This clearly implies the Parseval relation $\|f\|^2=\sum_\lambda|(f, g_\lambda)|^2$. The proof given here uses only Arzéla's theorem and elementary properties of completely continuous operators, and occupies 41 lines. *E. Hewitt* (Seattle, Wash.).

Takeda, Zirô, and Turumaru, Takasi. On the property "Position p". Math. Japonicae **2**, 195–197 (1952).

Various connections between the relation of being in "position p" for a pair of projections on a Hilbert space [see J. Dixmier, Revue Sci. **86**, 387–399 (1948); these Rev. **10**, 546], the notions of right and left projections [in the sense of I. Kaplansky, Ann. of Math. (2) **53**, 235–249 (1951); these Rev. **13**, 48], and equivalence in the sense of Murray and von Neumann [ibid. **37**, 116–229 (1936)] are obtained.

I. E. Segal (Chicago, Ill.).

Nakamura, Masahiro, and Turumaru, Takasi. Simple algebras of completely continuous operators. Tôhoku Math. J. (2) **4**, 303–308 (1952).

The algebras in question are uniformly closed self-adjoint operator algebras on Hilbert space, and simplicity is meant

in the sense of no closed two-sided ideals. The reviewer [Duke Math. J. 16, 399–418 (1949); these Rev. 11, 115] showed that such an algebra can be represented as all the completely continuous operators on a suitable Hilbert space. The authors' program goes beyond this to study all the representations, and it turns out that any representation is the direct sum of a finite number of equivalent irreducible ones. They proceed to study the commuting algebra A' of the given algebra A, and observe that if A is infinite-dimensional, A' degenerates to the annihilator of A. To remedy this they allow A' to consist of all commuting operators (not just the completely continuous ones), when A is infinite-dimensional. This leads to a satisfactory duality between A and A'. *I. Kaplansky* (Chicago, Ill.).

Turumaru, Takasi. On the direct-product of operator algebras. I. Tôhoku Math. J. (2) 4, 242–251 (1952).

The author gives a direct algebraic construction for the direct product of two C^*-algebras (with unit). An explicit crossnorm [see R. Schatten, A theory of cross-spaces, Princeton, 1950; these Rev. 12, 186] is given relative to which the product is again a C^*-algebra. *I. E. Segal.*

Takeda, Zirô. On a theorem of R. Pallu de la Barrier. Proc. Japan Acad. 28, 558–563 (1952).

The author gives two proofs of the following theorem ascribed by J. Dixmier [Summa Brasil. Math. 2, 151–182 (1951); these Rev. 14, 69] to R. Pallu de la Barrière: If E is a hyperstonian space and A is a commutative W^*-algebra on a Hilbert space H whose spectrum is E, then for any normal measure m on E, there exist vectors x and y in H with $\int_E A(p)dm(p)=(Ax, y)$. This can be deduced from the following direct consequence of multiplicity theory: a linear functional L on A with the property that $L(T_n)\to L(T)$ when $T_n \uparrow T$ has the form $L(A)=(Ax, y)$. The author gives two applications of de la Barrière's theorem, including a result previously proved by means of multiplicity theory, to the effect that any commutative W^*-algebra is isomorphic to a maximal abelian algebra. *I. E. Segal.*

Matsushita, Shin-ichi. Multiplicative linear functionals on B-algebras. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 3, 15–25 (1952).

The central result is apparently the known one that a B^*-algebra is commutative if and only if it has a complete set of homomorphisms into the complex field.

I. E. Segal (Chicago, Ill.).

Umegaki, Hisaharu. Operator algebra of finite class. Kōdai Math. Sem. Rep. 1952, 123–129 (1952).

The author treats the decomposition of central positive linear functionals on self-adjoint operator algebras for which the rings associated with the corresponding two-sided representation are finite. The results are applied to obtain a Plancherel type formula for locally compact groups with a complete system of neighborhoods invariant under inner automorphisms. *I. E. Segal* (Chicago, Ill.).

Wermer, John. Ideals in a class of commutative Banach algebras. Duke Math. J. 20, 273–278 (1953).

This paper deals with Banach algebras A which are commutative and semi-simple. The author shows that such an algebra, provided it satisfies certain conditions which are always valid in the group algebra of a locally compact group, has certain properties which had hitherto been established only for group algebras (by the author, I. Kaplansky, and H. Reiter). Letting M denote the space of maximal ideals in A, these conditions are as follows. (1) A is regular in the sense of Šilov. (2) $\{x \in A \mid x$ vanishes outside some compact subset of $M\}$ is dense in A. (3) Every hyperplane of A which is taken into itself by every $U \in G(A)$ is an ideal of A, where $G(A)$ denotes the family of linear isometries $U: A\to A$ with $Ux\cdot y=Uy\cdot x$ for every $x, y \in A$. (4) If $x\in A$ and $\epsilon>0$, then there exists a $y \in A$ with $\|xy-x\|<\epsilon$. The following are the main theorems of the paper. (A) Let A satisfy (1), (2), and (3). If I is a closed ideal in A contained in precisely one maximal ideal m_0, then $I=m_0$. (B) Let A satisfy (1), (2), (3), and (4). Then every closed ideal I in A, such that the boundary of $h(I)=\{m\in M \mid I\subset m\}$ has no perfect, non-empty subset, is the intersection of the elements of $h(I)$.

E. Michael (Seattle, Wash.).

Segal, I. E. A non-commutative extension of abstract integration. Ann. of Math. (2) 57, 401–457 (1953).

Soient $\mathfrak{H}$ un espace hilbertien, $\mathfrak{A}$ un anneau d'opérateurs dans $\mathfrak{H}$, $E\to m(E)$ une fonction positive finie ou non définie sur l'ensemble des projecteurs de $\mathfrak{A}$, complètement additive, unitairement invariante, et telle que tout projecteur de $\mathfrak{A}$ soit borne supérieure de projecteurs m-finis. Le système $\Gamma=(\mathfrak{H}, \mathfrak{A}, m)$ est appelé un "gage space", m étant le "gage". Exemple: soient R un espace mesuré, r la mesure; soit $\mathfrak{H}$ l'espace hilbertien des fonctions de carré intégrable sur R, $\mathfrak{A}$ l'algèbre des opérateurs de multiplication par les fonctions mesurables bornées; un projecteur E de $\mathfrak{A}$ est défini par la fonction caractéristique d'un ensemble mesurable $S\subset R$; posons $m(E)=r(S)$; alors $(\mathfrak{H}, \mathfrak{A}, m)$ est le gage space le plus général pour lequel $\mathfrak{A}$ est abélien. Ainsi la théorie des gage spaces est la "théorie non commutative de l'intégration". Pour construire cette théorie, l'auteur définit d'abord les opérateurs mesurables: ce sont les opérateurs T fermés, $T\,\eta\,\mathfrak{A}$, pour lesquels il existe une suite croissante de sous-espaces vectoriels fermés $\mathfrak{K}_n$ de $\mathfrak{H}$, tendant vers $\mathfrak{H}$, contenus dans l'ensemble de définition de T, et tels que $\mathfrak{H}\ominus\mathfrak{K}_n$ soit fini (m ne figure pas dans cette définition; de même, dans l'intégration ordinaire, seule la classe de la mesure influe sur la définition des fonctions mesurables). L'auteur étend à ces opérateurs les propriétés établies par Murray et von Neumann pour les opérateurs fermés d'un facteur de type II_1, et munit leur ensemble d'une structure de $*$-algèbre. Il définit la convergence presque partout d'une suite d'opérateurs mesurables, et établir les propriétés de continuité des opérations algébriques vis-à-vis de cette convergence. Les opérateurs intégrables sont introduits ensuite: on définit d'abord l'intégrale des opérateurs de $\mathfrak{C}$ (idéal des opérateurs $T\in\mathfrak{A}$ tels que $m(\overline{T(H)})<+\infty$, c'est-à-dire de "rang" m-fini) par extension de m; ceci permet de munir $\mathfrak{C}$ d'une norme, puis, par passage à la limite à partir de $\mathfrak{C}$, de définir les opérateurs intégrables et leur intégrale m, les espaces $L^1(\Gamma)$ et $L^2(\Gamma)$. On établit le théorème de Fischer-Riesz, la dualité entre L^1 et $L^\infty=\mathfrak{A}$, le théorème de passage à la limite monotone, des théorèmes de Radon-Nikodým; par exemple, si n est un autre gage sur $\mathfrak{A}$, on a, en gros, l'existence et l'unicité d'un opérateur autoadjoint positif $S\,\eta\,\mathfrak{C}$ (le centre de $\mathfrak{A}$) tel que $n(X)=m(XS)$ pour tout opérateur autoadjoint positif $X\,\eta\,\mathfrak{A}$.

Le reste du mémoire concerne des propriétés des anneaux d'opérateurs moins directement reliées à l'intégration. Soient $\mathfrak{A}$ un anneau d'opérateurs, φ un isomorphisme du centre $\mathfrak{C}$ de $\mathfrak{A}$ sur l'algèbre L^∞ des fonctions mesurables bornées d'un espace mesuré. Il existe une application $E\to d(E)$ définie sur l'ensemble des projecteurs de $\mathfrak{A}$, à

valeurs dans $(L^\infty)^+$, avec les propriétés suivantes: 1) $d(E)$ est finie localement presque partout si et seulement si E est fini; 2) si $EF=0$, $d(E+F)=d(E)+d(F)$; 3) d est complètement additive (au sens de la relation d'ordre classique dans L); 4) si $E\sim F$, $d(E)=d(F)$; 5) si E est un projecteur non nul de $\mathfrak{C}$, $d(E)\neq 0$ et $d(EF)=\varphi(E)d(F)$. L'auteur étudie les algébres hilbertiennes, introduit les éléments "bornés", les anneaux d'opérateurs $\mathfrak{L}$ et $\mathfrak{R}$ engendrés par les opérateurs de multiplication à gauche et à droite montre que $\mathfrak{L}=\mathfrak{R}'$, $\mathfrak{L}'=\mathfrak{R}$, $J\mathfrak{L}J=\mathfrak{R}$ (J étant l'involution définie par l'algèbre), $JTJ=T^*$ pour $T\in\mathfrak{L}\cap\mathfrak{R}$ (propriétés que l'auteur exprime en disant qu'on a un anneau d'opérateurs standard), et qu'il existe un gage m sur $\mathfrak{L}$ tel que $m(P)=\|x\|^2$ si P est l'opérateur de multiplication à gauche défini par l'élément borné x, $m(P)=+\infty$ dans le cas contraire [cf. Godement, J. Math. Pures Appl. (9) **30**, 1–110 (1951), en particulier p. 78; ces Rev. **13**, 12]. Réciproquement un gage space $\Gamma=(\mathfrak{H},\mathfrak{A},m)$ définit une algèbre hilbertienne (on prend les éléments de $\mathfrak{A}$ qui appartiennent à $L^2(\Gamma)$). Un anneau d'opérateurs peut être associé à une algèbre hilbertienne si et seulement s'il est standard et sans projecteurs purement infinis. Deux anneaux standard algébriquement isomorphes sont spatialement isomorphes. *J. Dixmier* (Dijon).

Calculus of Variations

Morrey, Charles B., Jr. Quasi-convexity and the lower semicontinuity of multiple integrals. Pacific J. Math. **2**, 25–53 (1952).

A function is said to be quasi-convex if and only if any linear function furnishes the absolute minimum among all Lipschitzian functions coinciding with it on the boundary. It is shown that for the lower semicontinuity of integrals of the form

$$I(z,D)=\int_D f(x,z,p)dx$$

$$x=(x^1,\cdots,x^\nu),\quad z=(z^1,\cdots,z^N),\quad p=p_\alpha^i$$

$$(i=1,\cdots,N,\ \alpha=1,\cdots,\nu)$$

with respect to various types of convergence of the vector function z, the quasi-convexity of $f(x,z,p)$ in p for each fixed (x,z) is a necessary and sufficient condition. Furthermore, if f is weakly quasi-convex (i.e., if and only if each linear function furnishes a weak relative minimum among all Lipschitzian functions coinciding with it on the boundary) and continuous, then f satisfies a uniform Lipschitz condition on any bounded set in p-space and satisfies a generalized Weierstrass condition which reduces to the ordinary Weierstrass condition if $f\in C'$ and is equivalent to the Legendre-Hadamard conditions if $f\in C''$. Some necessary conditions and a sufficient condition for the quasi-convexity of f are given. When f has certain special forms, conditions which are both necessary and sufficient are determined. The author apologizes for being unable to establish conditions which are both necessary and sufficient in the general case. *P. Nesbeda* (Camden, N. J.).

Fet, A. I. A connection between the topological properties and the number of extremals on a manifold. Doklady Akad. Nauk SSSR (N.S.) **88**, 415–417 (1953). (Russian)

Some corollaries are given to a previous paper of the author [Mat. Sbornik N.S. **30**(72), 271–316 (1952); these Rev. **13**, 955] concerning calculus of variations in the large for positive regular problems on a four times differentiable manifold R. Thus for a closed R whose groups $\Delta^r(R)$ are trivial for $1\leq r\leq k$ and $\Delta^{k+1}(R)\neq 0$, the number of closed extremals is not smaller than the $(k+1)$th Betti number. By an argument of Marston Morse [The calculus of variations in the large, Amer. Math. Soc. Colloq. Publ., v. 9, New York, 1934] and, in an alternate case, by one of B. A. Rohlin [Uspehi Matem. Nauk (N.S.) **1**, no. 5–6 (15–16), 175–223 (1946); these Rev. **10**, 393], for any non-simple closed R there are either at least three non-nulhomotopic closed extremals, or one non-nulhomotopic and one nulhomotopic closed extremals, or a family of nulhomotopic and one non-nulhomotopic closed extremals (thus always at least two closed extremals). *L. Cesari.*

Fet, A. I. On the algebraic number of closed extremals on a manifold. Doklady Akad. Nauk SSSR (N.S.) **88**, 619–621 (1953). (Russian)

Let R be a closed manifold whose $m-1$ first Betti groups (mod 2) vanish and whose mth Betti group contains at least one element of even order. The author's main result is that, given any positive regular variational problem for curves on R, the algebraic number of closed extremals is not less than 3; and further that there exist either continuum-many closed extremals of equal length, or else 3 closed extremals of indices $m-1$, $2(m-1)$, $3(m-1)$. This generalizes an earlier result in which R was assumed to be a 2-sphere [same Doklady (N.S.) **66**, 347–350 (1949); these Rev. **11**, 47]. The tools needed for this extension include a lower estimate of the "length" (mod 2), in the sense of Froloff and Elsgolz [Mat. Sbornik **42**, 637–642 (1935)], of the space of closed non-oriented curves on R. This "length", and also the Lusternik-Schnirelmann category, are shown to be not less than 3. *L. C. Young* (Madison, Wis.).

Theory of Probability

Isaacs, Rufus. Optimal horse race bets. Amer. Math. Monthly **60**, 310–315 (1953).

In a horse race it is assumed that the probabilities p_i for the ith horse to win is known as well as the amount $s_i>0$ wagered on each horse. By betting amounts x_i on the various horses the expected gain is found to be

$$F(x_1,\cdots,x_n)=Q\left[\sum_{j=1}^{n}(x_j+s_j)\right]\sum_{i=1}^{n}\frac{p_ix_i}{x_i+s_i}-\sum_{i=1}^{n}x_i$$

where $0<Q<1$ is a factor depending on the proportion of the total amount wagered which is returned to the bettors. The author solves the problem of maximizing F and shows that there is a positive maximum only if for some j

$$\frac{p_j}{s_j}>\frac{1}{Q\sum_{i=1}^{n}s_i}.$$

The problem is of theoretical interest since it affords a nonlinear programming problem which can be solved explicitly. *O. Ore* (New Haven, Conn.).

Aoyama, Hirojiro. On Midzuno's inequality. Ann. Inst. Statist. Math., Tokyo **3**, 65–67 (1952).

Under certain regularity hypotheses concerning the density f of a chance variable x, the inequality

$$\text{Prob}\,\{|x-Ex|\geq k\sigma\}\leq\frac{1\cdot3\cdots(2\lambda-1)}{k^\lambda}(1+g(\lambda,C))$$

is obtained, where λ is a positive integer,

$$C=\int_{-\infty}^{\infty} f''(x) \exp [x^2/4] dx,$$

assumed finite, and $g(\lambda, C)=1.09C\sum_2^{\lambda}(2^j j!)^2\binom{\lambda}{j}/(2j+1)!$. The inequality is a correction to that of Midzuno [same Ann. 2, 21–33 (1950); these Rev. 12, 509], who neglected $g(\lambda, C)$. *D. Blackwell* (Washington, D. C.).

Smith, Walter L. On the distribution of queueing times. Proc. Cambridge Philos. Soc. **49**, 449–461 (1953).

A detailed function-theoretic study is made of the integral equation derived by Lindley [same Proc. **48**, 277–289 (1952); these Rev. **13**, 759] for the distribution of waiting time before a single server when both service time and time between arrivals are of a general character. The main interest is in determining in what circumstances this distribution may be simple. It is shown that the character of the service-time distribution is strongly reflected in the character of the distribution of the sum of service and waiting time (the time interval between arrival and departure); for exponentially distributed service times the sum of service and waiting times must also be exponentially distributed, whatever the character of incoming traffic in a wide class of traffics. A number of interesting applications are made. *J. Riordan* (New York, N. Y.).

Prékopa, András. On composed Poisson distributions. IV. Remarks on the theory of differential processes. Acta Math. Acad. Sci. Hungar. **3** (1952), 317–325 (1953). (Russian summary)

[For parts I–III see these Rev. **13**, 663; **14**, 770.] Let ξ_t be a stochastic process defined on a finite interval I. Let ξ_J denote the difference $\xi_{t_2}-\xi_{t_1}$ where J is a subinterval (t_1, t_2) of I, and $W_\lambda(J)$ denotes $\Pr\{\xi_J=\lambda\}$. The following conditions are assumed: A) ξ_J is a process with independent increments; B) ξ_J takes on values in a fixed countable set, $\lambda_0=0, \lambda_1, \lambda_2, \cdots$; C) $1-W_0(J)\rightarrow 0$ when J decreases to a fixed point. Under these conditions it is proved that the logarithm of the characteristic function of ξ_J can be written in the form $\log f(u, I)=\sum_{k=1}^{\infty} C_{\lambda_k}(I)(e^{i\lambda_k u}-1)$, where

$$C_{\lambda_k}(I)=\int_I W_{\lambda_k}(J) \quad (\lambda_k\neq 0)$$

and

$$\sum_{k=1}^{\infty} C_{\lambda_k}(I)=\int_I (1-W_0(J))<\infty.$$

The integrals are understood in the sense of Burkill. The theorem is first proved using the normal form of an infinitely divisible distribution and then proved without using this normal form by means of a theorem on almost periodic functions. *J. L. Snell* (Princeton, N. J.).

Milicer-Grużewska, Halina. Sur la répartition des deux variables aléatoires dépendantes. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **43** (1950), 98–120 (1952).

If $F_1(x)$ is a distribution function (d. f.; defined to be right continuous) and $F_x(y)$ is a d. f. in y for each x and right continuous in x for each y, then $F(x, y)$ defined by $\int_{-\infty}^{x} F_t(y)dF_1(t)$ is a d. f. such that

$$\lim \Delta_x F(x, y)/\Delta F_1(x)=F_x(y)$$

is a d. f. in y at each point of increase of $F_1(x)$. Furthermore, if $F(y)=F(\infty, y)$, then $\bar{F}_y(x)=\limsup \Delta_y F(x, y)/\Delta F(y)$ is a d. f. in x at each point of increase of $F(y)$ if and only if it is right continuous, but always $F(x, y)=\int_{-\infty}^{y} \bar{F}_t(x)dF(t)$. This last follows from a classical theorem of Lebesgue. Conditions are given for a given function $\mathfrak{F}_x(y)$ to be such that there exists a d. f. $F(x, y)=F_1(x)\mathfrak{F}_x(y)$ for a given (or for every) d. f. $F_1(x)$. *K. L. Chung* (Ithaca, N. Y.).

Sugiyama, Hiroshi. On the asymptotic behavior of $\sum p_m^2$ in case of certain probability distributions. I. Math. Japonicae **2**, 187–192 (1952).

The author considers a previous paper by R. M. Redheffer [Ann. Math. Statistics **22**, 128–130 (1951)] and adds some remarks. *K. L. Chung* (Ithaca, N. Y.).

Ríos, Sixto. Some probability laws and stochastic processes deduced from a Laplace-Stieltjes integral. Revista Acad. Ci. Madrid **46** (1952), 487–490 (1953). (Spanish)

For an exponential family of distributions, i.e., a family with density $f(\theta)e^{\theta x}$ with respect to a measure μ, the characteristic function is $\phi_s(t)=f(s+it)/f(s)$, so that the moments are easily calculable from f. Several instances of exponential families are listed. *D. Blackwell* (Washington, D. C.).

Mattila, Sakari. On biorthogonal expansions of the conjugate random functions. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **143**, 8 pp. (1952).

The author establishes a biorthogonal expansion of conjugate random functions defined on a segment $[a, b]$; Smithies' extension [Proc. London Math. Soc. (2) **43**, 255–279 (1937)] of Mercer's theorem is used. The result extends one by the reviewer [C. R. Acad. Sci. Paris **222**, 469–470 (1946); these Rev. **7**, 458], also by Karhunen [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **37** (1947); these Rev. **9**, 292]. *M. Loève* (Berkeley, Calif.).

Fortet, Robert, et Mourier, Edith. Convergence de la répartition empirique vers la répartition théorique. C. R. Acad. Sci. Paris **236**, 1739–1740 (1953).

The authors state without proof two theorems which, in the interests of brevity, we cite for the real line. Let $\{X_n\}$ be a sequence of independent chance variables with the same distribution function $F(x)$. Let $f(x)$ be a point function, $M(f)=(\sup|f(x_1)-f(x_2)|)/|x_1-x_2|$, and $N(f)=\sup|f(x)|$. Let $C_1(\rho)$ be the class of functions f such that $M(f)\leqq\rho$, and let $C_2(\rho)$ be the class of functions f for which $M(f)+N(f)\leqq\rho$. I) Let ρ be any positive number, x_0 be any point, and $F(x)$ be such that $\int|x-x_0|dF(x)<\infty$. With probability one (w.p.1) $n^{-1}\sum_{j=1}^{n} f(X_j)=a_n$ approaches $\int f(x)dF(x)=b$, uniformly for all f in $C_1(\rho)$. II) W.p.1 for any positive ρ, $a_n\rightarrow b$ uniformly for all f in $C_2(\rho)$. *J. Wolfowitz.*

Mulholland, H. P. An inequality related to the central limit theorem on probabilities. J. London Math. Soc. **28**, 360–369 (1953).

The distribution function (d.f.) of a chance variable X is said to be more peaked about 0 than that of Y if $P\{|X|\leqq x\}\geqq P\{|Y|\leqq x\}$ for every $x\geqq 0$ [Birnbaum, Ann. Math. Statistics **19**, 76–81 (1948); these Rev. **9**, 452]. The author calls a d.f. "unimodal" if it is the integral of a density function that is monotonic on each side of its mode. He calls a d.f. $P(x)$ with mean zero and variance s^2 "subnormal" if $\int e^{ux}dP(x)\leqq\exp\{s^2u^2/2\}$ for all real u. Several results on the peakedness of the d.f. are proved, of which the following is perhaps the principal: Let $\{x_i\}$ be a sequence of independent random variables with respective variances $\{s_i^2\}$ and with symmetrical, unimodal, and subnormal d.f.'s.

Then the d.f. of $\sum_{i=1}^{n} x_i$ is more peaked about zero than the normal d.f. with mean zero and variance σ_n^2 if $\sigma_1^2 \geqq 6s_1^2/\pi$, $\sigma^2_{n+1} \geqq \sigma_n^4/(\sigma_n^2 - s^2_{n+1})$ for $n=1, 2, \cdots$, and also if

$$\sigma_n^2/S_n^2 \geqq 1 + R_n^{-1} \log (e^2 R_n)$$

where $S_n^2 = s_1^2 + \cdots + s_n^2$ and $R_n = \{S_n/\max (s_1, \cdots, s_n)\}^2$.
J. Wolfowitz (Ithaca, N. Y.).

Lévy, Paul. Loi faible et loi forte des grands nombres. Bull. Sci. Math. (2) 77, 9–40 (1953).

According to a footnote this paper will form a new Note to the revised edition of the author's Théorie de l'addition des variables aléatoires [Gauthier-Villars, Paris, 1937]. It summarizes some known results and adds several new ones, achieving a notable synthesis of the laws of large numbers and the central limit theorem. The main results have been announced and reviewed [C. R. Acad. Sci. Paris **235**, 1186–1188 (1952); these Rev. **14**, 485]; here are further details and relevant remarks. The classification according to the Laplacian character of the X_ν is given another, more manageable, form as follows. Suppose the X_ν have median 0 and let X_ν' be the truncated variables at $\pm l$, σ_n' = the standard deviation of $\sum_1^n X_\nu'$; $\eta_n = l/\sigma_n'$, $\eta_n' = \sum_1^n \Pr (|X_\nu| > l)$; $\epsilon_n = \inf (\eta_n + \eta_n')$ for varying l. One is in the Laplacian, intermittently Laplacian or non-Laplacian case according as $\lim \epsilon_n = 0$, $\limsup \epsilon_n > \liminf \epsilon_n = 0$ or $\liminf \epsilon_n > 0$.

Theorem 1, that (1_f) implies (2_f), was actually first proved by Kolmogorov [Math. Ann. **102**, 484–488 (1929); see (16)]. Although he stated it only for the case $a_n = n$ the proof needs no change. It is of course also a consequence of Feller's theorem (Theorem 3 here). However, the author derives it anew from the neat inequality:

$$\Pr (|S_n - m_n| \geqq a_n/2) \geqq \Pr (M_n > a_n)/6.$$

Now he notices that the converse is true, namely (2_f) implies (1_f), in the non-Laplacian case.

For the strong versions, it is easy to prove that (1_F) implies (2_F), but the converse implication requires a stronger condition, that of "equally non-Laplacian": there exists a constant c such that $\int_0^x y^2 |dF_\nu(y)| \leqq cx^2 F_\nu(x)$ for all ν and all (large) x. Under this condition an inequality is obtained which sort of reverses the one above:

$$\Pr (|S_n - \mu_n'| > 2l) < C \Pr (M_n > l)$$

where μ_n' is some constant and C depends only on the c involved in the definition above and not on the arbitrary l. This inequality and the technique of extraction of subsequences suffice to prove the "fundamental theorem" that (2_F) implies (1_F).

Another result: If the X_ν are equally non-Laplacian, then for every $\epsilon > 0$ there corresponds a finite $K = K(c, \epsilon)$ and $\mu_n' = \mu_n'(\epsilon)$ such that $\Pr (|S_n - \mu_n'| > KM_n) < \epsilon$. This result and some examples regarding the relative magnitudes of $S_n - m_n$ and M_n are related to some recent results of Darling [Trans. Amer. Math. Soc. **73**, 95–107 (1952); these Rev. **14**, 60]. The latter considers more specific (non-Laplacian) cases and obtains more specific results. *K. L. Chung.*

Chung, Kai Lai. On the renewal theorem in higher dimensions. Skand. Aktuarietidskr. **35** (1952), 188–194 (1953).

For any sequence of independent r-dimensional identically distributed chance variables $V_1, V_2, \cdots$ with $r > 1$, any bounded set C of r-space and any vector U in r-space, the expected number of sums $V_1 + \cdots + V_n$ lying in $C + U$, the set C translated by U, approaches zero as $|U| \to \infty$.
D. Blackwell (Washington, D. C.).

Blackwell, David. Extension of a renewal theorem. Pacific J. Math. **3**, 315–320 (1953).

A chance variable x is called a d-lattice variable if $\sum_{n=-\infty}^{n=+\infty} \Pr \{x = nd\} = 1$ and d is the largest number for which this is true. If x is not a d-lattice variable for any d, x, it is called a nonlattice variable. The author proves the following theorem: Let $x_1, x_2, \cdots$ be independent identically distributed chance variables with $E(x_1) = m > 0$ ($+\infty$ admitted); let $S_n = x_1 + x_2 + \cdots + x_n$; and, for any $h > 0$, let $U(a, h)$ be the expected number of integers $n \geqq 0$ for which $a \leqq S_n < a + h$. If the x_n are nonlattice variables, then $U(a, h) \to h/m$, 0 as $a \to +\infty, -\infty$. If the x_n are d-lattice variables, then $U(a, d) \to d/m$, 0 as $a \to +\infty, -\infty$. As the author indicates, this theorem combines the recent extensions in the literature of the theorem for nonnegative d-lattice variables which is of basic importance in the theory of Markov chains with enumerably many states. The theorem is shown to follow from this special case by using an integral identity and Wald's equation from the theory of sequential analysis. *J. L. Snell* (Princeton, N. J.).

Steinhaus, H. Sur les fonctions indépendantes. VIII. Studia Math. **11**, 133–144 (1949).

[For part VII see Studia Math. **10**, 1–20 (1948); these Rev. **9**, 292.] The paper consists of a number of remarks on the strong law of large numbers. A typical result is the following: Let $x_i(t)$ be measurable functions on (0, 1) which are independent in pairs. Let $F_i(\alpha)$ be the distribution function of $x_i(t)$ and assume that

$$\lim_{n\to\infty} \frac{1}{n} \sum_1^n F_i(\alpha) = F(\alpha)$$

exists for every real α. Let furthermore $\psi_\alpha(x)$ be 1 or 0 according as $x < \alpha$ or $x \geqq \alpha$. Then there exists a set $T \subset (0, 1)$ of measure one such that for every $t \in T$ and every real α

$$\lim_{n\to\infty} \frac{1}{n} \sum_1^n \psi(x_i(t)) = F(\alpha).$$

M. Kac (Ithaca, N. Y.).

Ryll-Nardzewski, C. D. Blackwell's conjecture on power series with random coefficients. Studia Math. **13**, 30–36 (1953).

For any sequence F of independent random variables $x_1, x_2, \cdots$, denote by $R(F)$ the radius of convergence of the series $\sum x_n z^n$ ($R(F)$ is constant with probability one). It is shown that for every F there is a sequence of constants a_n such that: (1) for every sequence of constants b_n, $R\{x_n + b_n\} \leqq R\{x_n + a_n\}$; (2) $\sum (x_n + a_n) z^n$ is singular on its circle of convergence; and (3) $R\{x_n + b_n\} = R\{x_n + a_n\}$ implies that $\sum (x_n + b_n) z^n$ is singular on its circle of convergence.
D. Blackwell (Washington, D. C.).

Wasow, Wolfgang. On the duration of random walks. Ann. Math. Statistics **22**, 199–216 (1951).

Continuing an earlier investigation [J. Research Nat. Bur. Standards **46**, 462–471 (1951); these Rev. **13**, 960], the author shows how to determine the moment-generating function of the duration of fairly general random walks in a region. As an application he obtains explicit expressions for the variance of duration and in some cases estimates of probabilities of extremely long walks. The method is essentially the same as used in the paper cited above.
M. Kac (Ithaca, N. Y.).

King, Gilbert W. **Further remarks on stochastic methods in quantum mechanics.** Proceedings, Computation Seminar, December 1949, pp. 92–94. International Business Machines Corp., New York, N. Y., 1951.

Amplification of an earlier paper [Proceedings, Seminar on Scientific Computation, 1949, International Business Machines, New York, 1950, pp. 42–48; these Rev. **14**, 414; in this review "random values" should read "random walks"]. *M. Kac* (Ithaca, N. Y.).

Kampé de Fériet, Joseph. Autocorrélation et spectre quadratique d'une fonction définie sur un groupe abélien localement compact. C. R. Acad. Sci. Paris **236**, 2198–2200 (1953).

The author presents and discusses an extension of the notion of autocorrelation of a function to the case of suitable functions square-integrable on every compact subset of a locally compact abelian group. *I. E. Segal.*

Bunimovič, V. I. The fluctuation process as an oscillation with random amplitude and phase. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 1231–1259 (1949). (Russian)

Lehan, Frank W. Expected number of crossings of axis by linearly increasing function plus noise. J. Appl. Phys. **22**, 1067–1069 (1951).

Mathematical Statistics

Davidoff, Melvin D., and Goheen, Howard W. A table for the rapid determination of the tetrachoric correlation coefficient. Psychometrika **18**, 115–121 (1953).

Chandra Sekar, C., and Chakraborty, P. N. On the concept and use of orthogonal semi-polynomials. Sankhyā **12**, 141–150 (1952).

A semi-polynomial is one that has a different intercept for different parts of the data. In this article the p observations are divided into two parts and represented by the polynomial

$$Y = z(x) + \sum_{i=1}^{m} b_i \psi_i(x) + \text{error},$$

where

$$z(x) = \begin{cases} z(B) \text{ for the first } n_1 \text{ observations} \\ z(C) \text{ for the last } (p-n_1) \text{ observations,} \end{cases}$$

and ψ_i is an ith degree polynomial. Methods of determining normal-orthogonal ψ_i are presented. Then the method of least squares is used to derive simple estimates of $z(x)$, the b_i, and standard errors of the b_i. Hence it is possible to test for additional terms in the polynomial by the usual methods. *R. L. Anderson* (Raleigh, N. C.).

Sampford, M. R. Some inequalities on Mill's ratio and related functions. Ann. Math. Statistics **24**, 130–132 (1953).

For Mill's ratio defined as $R_x = e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}u^2}du$, the inequalities $\frac{1}{2}\{(4+x^2)^{1/2}-x\} < R_x < 1/x$, $x>0$, were known. For the functions $\nu(x) = 1/R_x$, $\lambda(x) = \nu'(x) = \nu(\nu - x)$, the inequality $\lambda' > 0$ was conjectured by the reviewer for $x>0$ [same Ann. **21**, 272–279 (1950); these Rev. **11**, 673] and was proved by Murty for x sufficiently large [J. Indian Soc. Agric. Statistics **4**, 85–87 (1952); these Rev. **14**, 389]. In the present paper the author gives concise proofs of the inequalities $0<\lambda<1$ and $\lambda'>0$ for all finite x. *Z. W. Birnbaum* (Seattle, Wash.).

Tate, Robert F. On a double inequality of the normal distribution. Ann. Math. Statistics **24**, 132–134 (1953).

For Mill's ratio [see the preceding review] the author derives an upper bound which is an improvement on that given by Gordon [same Ann. **12**, 364–366 (1941); these Rev. **3**, 171] for $X \geqq 0$, and a lower bound which for some values $X \geqq 0$ is greater and for some smaller than a bound given by the reviewer [ibid. **13**, 245–246 (1942); these Rev. **4**, 19]. Similar bounds for $X<0$ are an immediate consequence. *Z. W. Birnbaum* (Seattle, Wash.).

Gayen, A. K. The inverse hyperbolic sine transformation on Student's t for non-normal samples. Sankhyā **12**, 105–108 (1952).

For normal samples the distribution of

$$y = \pm\sinh^{-1}(3t^2/2n),$$

where t is Student's t and n the corresponding number of degrees of freedom, approaches normality much more rapidly than t itself as n increases [cf. Anscombe, J. Roy. Statist. Soc. Ser. A. **113**, 228–229 (1950); these Rev. **12**, 207]. From a study of moment functions the author finds the use of y not advisable for non-normal samples. *D. M. Sandelius* (Uppsala).

Lieblein, Julius. On the exact evaluation of the variances and covariances of order statistics in samples from the extreme-value distribution. Ann. Math. Statistics **24**, 282–287 (1953).

Explicit closed formulas involving only tabulated functions are obtained for the covariances of order statistics based on samples of size n from the extreme-value distribution. The main point of the paper is the evaluation of the double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{y} xy \exp(-x - te^{-x} - y - ue^{-y})\,dx\,dy.$$

The procedure is illustrated by computations for $n=3$. *D. M. Sandelius* (Uppsala).

Tiago de Oliveira, J. A note on a special case of inverse binomial sampling. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. **2**, 111–114 (1952).

Let p be the proportion of some attribute A in a population which is sampled without replacement until on the nth trial an item with the attribute A is drawn. The author studies methods of estimating p and confidence intervals for p via the Tchebycheff inequality. A better solution is given by Craig [Industrial Quality Control **9**, 83–86 (1953)]. Results are extended to two populations. Misprints are disturbing. *L. A. Aroian* (Culver City, Calif.).

Tiago de Oliveira, J. Tests for the equality of proportions in a multinomial population. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. **2**, 197–200 (1952).

Given a population with three attributes A_1, A_2, and A_3 in proportions p_1, p_2, p_3, $p_1+p_2+p_3=1$, the author provides tests of the hypothesis $p_1=p_2=p$, p unknown in samples of n, when the number n_1 of A_1, and n_2 of A_2 are small. In case p is known and equal to p_0, he finds the distribution of $\delta = n_1 - n_2$ by means of recurrence relations and suggests this technique as a method of testing the hypothesis $p=p_0$. *L. A. Aroian* (Culver City, Calif.).

Madow, William G. On the theory of systematic sampling. III. Comparison of centered and random start systematic sampling. Ann. Math. Statistics **24**, 101–106 (1953).

An explicit formula is obtained for the difference in variances between the mean of a random-start systematic sample and that of a systematic-start random sample. It follows immediately from the formula that for populations with monotone decreasing correlogram, the centered-start mean has a smaller variance than does the random-start mean. *D. Blackwell* (Washington, D. C.).

Maritz, J. S. Estimation of the correlation coefficient in the case of a bivariate normal population when one of the variables is dichotomized. Psychometrika **18**, 97–110 (1953).

Basu, D. An example of non-existence of a minimum variance estimator. Sankhyā **12**, 43–44 (1952).

There are many familiar, simple examples of the non-existence of a uniform minimum variance unbiased estimator (e.g., in estimating the mean of a uniform distribution from θ to $\theta+1$ where $-\infty<\theta<\infty$). The author gives a more complicated one. *J. Kiefer* (Ithaca, N. Y.).

Yamamoto, Sumiyasu. On the estimation of the coefficient of variation by the ratio of two quantities in large samples. Kōdai Math. Sem. Rep. **1952**, 115–122 (1952).

For the problems of point and interval estimation of, and testing hypotheses about, the coefficient of variation V of a normal distribution, the author suggests certain procedures based on the ratio of two sample quantiles. He suggests that these procedures may be more convenient in some applications than classical ones based on the sufficient statistic, computes approximately optimum choices of the quantiles for the case when V is known to be $\leqq 0.1$, and shows that the efficiency of his procedure compared to the classical procedure (as measured by expected length of confidence interval) for this case is 0.8. Tables and charts are given to facilitate the use of these methods. *J. Kiefer.*

Taylor, William F. Distance functions and regular best asymptotically normal estimates. Ann. Math. Statistics **24**, 85–92 (1953).

Let p_θ be for each θ a distribution on a finite sample space, and let $\delta(p, q)$ be a distance function defined for all pairs p, q of distributions on the sample space. A technique for estimating θ based on a sample is to choose θ to minimize $\delta(p_\theta, q^*)$, where q^* is the observed sample distribution. Under certain restrictions on δ and p_θ, the technique yields regular best asymptotically normal estimates. Special cases are estimates developed by Neyman and Berkson for multinomial and logistic distributions. *D. Blackwell.*

Kudō, Hirokichi. On a formulation of classical problems of statistics. Nat. Sci. Rep. Ochanomizu Univ. **1**, 9–13 (1951).

Let P be a fixed probability measure on a Borel field $\mathfrak{B}$ of subsets of a space Ω, let $\{\sigma_s\}$ be a group of transformations of Ω, $-\infty<s<\infty$, and write $P_s(E)=P(\sigma_s^{-1}E)$. If the class of likelihood ratio tests for P_{s_1} against P_{s_2} is independent of s_1, s_2, the group σ_s is called dissipative with respect to P. Some properties, e.g., regularity, differentiability, convexity for $s>0$ of the power function for likelihood ratio tests in dissipative groups, are announced. For estimating the parameter s, the author proposes an α-estimate, defined by $s_\alpha(w)=\sup\{s\,|\,w\in\sigma^s R_\alpha\}$, where R_α is the likelihood-ratio critical region having P-measure α. Some properties of α-estimates are announced, among which are consistency, sufficiency, and the translation property $s_\alpha(\sigma^s w)=s_\alpha(w)+s$. Certain choices of α yield maximum likelihood estimates. All families which are dissipative for all sample sizes are of Koopman type. The related work of Pitman [Biometrika **30**, 391–421 (1939)] and Girshick and Savage [Proc. Second Berkeley Symposium on Math. Statistics and Probability, 1950, Univ. of California Press, 1951, pp. 53–73; these Rev. **13**, 571] on estimation of location parameters is not mentioned. *D. Blackwell* (Washington, D. C.).

Kudō, Hirokichi. A remark on the efficient estimation. Nat. Sci. Rep. Ochanomizu Univ. **2**, 18–24 (1951).

For a parametric family of distributions, conditions are given under which there exists a unique 1-1 function of the parameter which has an efficient unbiased estimate. *D. Blackwell* (Washington, D. C.).

Sakaguchi, Minoru. Notes on statistical applications of the information theory. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **1**, no. 4, 27–31 (1952).

If $x_1, x_2, \cdots$ are independent chance variables with a common distribution F and $s=\{S_k\}$ is a sequential sampling plan, i.e., a sequence of mutually exclusive events with S_k depending only on $x_1, \cdots, x_k$ and $\sum_1^\infty P(S_k)=1$, it follows from Wald's equation [Ann. Math. Statistics **16**, 117–186 (1945); these Rev. **7**, 131] that the entropy $H(s, F)$ of the sequential sample is related to the entropy $H(F)$ of a single observation by (1) $H(s, F)=E(n)H(F)$. A direct proof of (1) is given. For a fixed s and a class C of distributions F for which $E(n)$ is fixed, maximizing $H(s, F)$ over C is by (1) equivalent to maximizing $H(F)$ over C. Three examples are given, one of which is: $S_k=\{x_1+\cdots+x_k\geqq a, x_1+\cdots+x_j<a$ for $j<k\}$ and C consists of all absolutely continuous distributions on 0, ∞ with mean μ. For large a, $E(n)\sim a/\mu$, and Max $H(F)$ occurs when F has density $\mu^{-1}\exp[-x/\mu]$. *D. Blackwell* (Washington, D. C.).

Rushton, S. On sequential tests of the equality of variances of two normal populations with known means. Sankhyā **12**, 63–78 (1952).

A theorem of Barnard [Biometrika **39**, 144–150 (1952); these Rev. **14**, 65] is applied to determine two sequential tests for the equality of variances of two normal populations, with known means. One of the tests is based on the range, the other on the usual F-statistic. The latter test is also extended to the case where the means are unknown. The tests depend on reducing the original composite hypothesis by invariance considerations to a simple hypothesis concerning the distribution of the F-statistic. Tables are given to facilitate application of the tests but there is no evaluation of the average sample size. *D. G. Chapman.*

Hamaker, H. C. The efficiency of sequential sampling for attributes. I. Theory. Philips Research Rep. **8**, 35–46 (1953).

The author gives approximate calculations of the OC and ASN curves for five possible choices of a Wald sequential procedure in a case of sampling for attributes, in order to support his contention that among all such procedures, which may be characterized by acceptance number $a_n=-a+ns$ and rejection number $b_n=b+ns$ after n observations, it suffices for practical applications to consider only procedures with $a=b$. *J. Kiefer* (Ithaca, N. Y.).

Kitagawa, Tosio. Successive process of statistical inferences. III. Mem. Fac. Sci. Kyūsyū Univ. A. 6, 131–155 (1952).

[For parts I and II see same Mem. 5, 139–180 (1950); 6, 55–95 (1951); these Rev. 13, 854; 14, 390.] The author continues his development of two and three sample theories where the data from one or two samples are used to make inferences about parameters or an unobserved third sample. Several applications to problems of estimation and prognosis are made. Finally the author presents his theory from a decision function point of view. *H. Chernoff.*

Kitagawa, Tosio. Successive process of statistical inferences. IV. Bull. Math. Statist. 5, 35–50 (1952).

A modified t-test is constructed involving the mean value of the ranges of several samples. Similar methods are applied to the successive poolings of data in control charts. It is claimed that Fisher's theory of fiducial inference applied to the Behrens-Fisher test can be more accurately and more thoroughly formulated from the two-sample theory point of view. The author illustrates what he means with a method and results similar to those of Barnard and Stein [Barnard, Biometrika 37, 203–207 (1950); these Rev. 13, 260; Stein, Ann. Math. Statistics 16, 243–258 (1945); these Rev. 7, 213]. *H. Chernoff* (Stanford, Calif.).

Kitagawa, Tosio. Successive processes of statistical controls. I. Mem. Fac. Sci. Kyūsyū Univ. A. 7, 13–28 (1952).

The problem of statistical controls arises in control chart methods but the literature does not have an adequate formulation involving successive processes of controls. This problem is formulated here from a decision function point of view. Results are obtained on the effects of certain techniques of successive regulations (adjustments). In one of these techniques an adjustment is made after each batch of observations and in another an adjustment is made only after a test of control is rejected. *H. Chernoff.*

Bahadur, Raghu Raj, and Goodman, Leo A. Impartial decision rules and sufficient statistics. Ann. Math. Statistics 23, 553–562 (1952).

Let $\pi_1, \cdots, \pi_k$ be populations with unknown distributions $F_1, \cdots, F_k$, and let u_i be a sample of n independent observations from π_i. A decision rule is a function

$$p(u)=(p_1(u_1, \cdots, u_k), \cdots, p_k(u_1, \cdots, u_k)), \quad p_i \geqq 0, \quad \sum_1^k p_i=1,$$

where $p_i(u)$ is the fraction of future observations to be drawn from π_i when u is observed. The problem is to select a decision rule so that the distribution $E\sum_1^k p_i F_i$ of future observations has some desirable property, e.g., that its mean is as large as possible. A decision rule is impartial if a permutation of the u_i leads to the same permutation of the p_i. It is shown that for any impartial decision rule, its expectation, given a sufficient statistic, is also impartial. An application is given to an extension of the problem of choosing which of k normal populations has the greatest mean. *D. Blackwell* (Washington, D. C.).

Sverdrup, Erling. Weight functions and minimax procedures in the theory of statistical inference. Arch. Math. Naturvid. 51, 117–192 (1952).

The paper consists partly of an exposition of the principles of decision theory, including an explicit statement of some general properties of minimax solutions, e.g., that a Bayes solution of constant risk is minimax, which, as the author surmises, are generally "known to statisticians who have been working on minimax problems". Minimax solutions of several familiar testing-hypotheses problems involving normal distributions are found; the minimax solutions turn out to be special cases of the pre-decision theory solutions of these problems: e.g., for testing $N(\theta, \sigma^2)$ vs $N(\theta, \tau^2)$, $\sigma^2 \leqq \sigma_1^2 < \sigma_2^2 \leqq \tau^2$, σ_1, σ_2 known, θ, σ, τ unknown, with constant loss for wrong decisions, the minimax test is to accept $\sigma^2 \leqq \sigma_1^2$ if $\sum(x_i-\bar{x})^2 \leqq k$ for an appropriate k.

The point is made that the property of being minimax is not particularly appealing unless it is accompanied by admissiblity; e.g., a minimax test of Student's hypothesis, with constant losses for wrong decisions, can be performed by ignoring the observations entirely; it is shown that Student's test at an appropriate level is admissible as well as minimax.

The final section makes a useful distinction between statistical inference (estimation and testing hypotheses), where the problem is to make a judgment about certain unknown parameters, and prediction, where the problem is to predict and control the values of unobserved chance variables whose distribution depends on the unknown parameters. It is noted that the ordinary two-step process, in which the statistician first makes an inference about a parameter, then takes that action or makes that prediction which would be best if his inference were correct, may be inefficient, unless the type of prediction to be made is taken into account in choosing the method of inference. A careful formulation of prediction problems is given, and it is shown that every prediction problem is reducible to an inference problem. *D. Blackwell* (Washington, D. C.).

Dvoretzky, A., Kiefer, J., and Wolfowitz, J. Sequential decision problems for processes with continuous time parameter. Testing hypotheses. Ann. Math. Statistics 24, 254–264 (1953).

A statistician observes a stochastic process $\{x(t), t \geqq 0\}$ known to be one of two possible stochastic processes and wishes to decide which of the two the given process is. The authors carry over for this problem with continuous time the well-known formulation of the corresponding problem with discrete time given by A. Wald. The processes are assumed to be stationary ~~with~~ independent increments, corresponding to independence with common distribution in the discrete time case. Under other such natural assumptions the authors define the probability ratio test for continuous time and indicate that the results for the discrete time case such as the optimal character of the ratio test, Wald's fundamental identity, etc., go over without new proofs. The authors study in detail special cases of the theory. As the authors point out, in some respects the continuous time theory is simpler and more elegant than the discrete time theory. For example, in the case of a Wiener process, the very simple formulas for defining the boundaries for a test of a given power, which were given by Wald as approximations in the discrete case, become in the continuous case exact, the reason being that the sample functions are continuous and hence there is no overlap at the boundaries when stopping sampling. In the case that the process being observed is one of two possible Poisson processes the authors show how to determine the boundaries for the ratio test of a given power and give the power function of the test, and the moment generating function of the observation time. *J. L. Snell* (Princeton, N. J.).

Wolfowitz, J. The method of maximum likelihood and the Wald theory of decision functions. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 114–119 (1953).

The author gives an explanation of why the maximum likelihood estimator is asymptotically efficient when it is. The argument is that for a certain problem in decision theory a complete class of estimators consists of Bayes solutions and that asymptotically every Bayes solution for suitable frequency function becomes a maximum likelihood estimator. The discussion is heuristic throughout but a method of making it rigorous is indicated. Neyman and Scott [Econometrica 16, 1–32 (1948); these Rev. 9, 600] showed that for a certain type of problem the maximum likelihood estimator is not asymptotically efficient and the author gives a heuristic explanation again based on decision theory of their results. *J. L. Snell* (Princeton, N. J.).

LeCam, Lucien. On some asymptotic properties of maximum likelihood estimates and related Bayes' estimates. Univ. California Publ. Statist. 1, 277–329 (1953).

The author proves that, within the set of asymptotically normal estimates, there is no estimate whose asymptotic variance is an absolute minimum. Under certain regularity conditions, the set S_g of points (in the space of the parameter θ) where an estimate g has asymptotic variance less than that of the maximum likelihood (m.l.) estimate has Lebesgue measure zero. Under certain conditions, if S_g is not empty, then there must be infinitely many points in the parameter space where the m.l. estimate has smaller asymptotic variance than g.

It is shown that, under certain conditions, m.l. estimates are asymptotically sufficient in a reasonable sense, and, whatever the a priori distribution of θ, its a posteriori distribution approaches the normal distribution as the number of observations increases; these ideas are also to be found in the paper reviewed above. *J. Wolfowitz.*

Elfving, G. Sufficiency and completeness in decision function theory. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **135**, 9 pp. (1952).

Let $m_1, \cdots, m_N$ be probability measures on a finite sample space, and let x, y be any two chance variables on this sample space. If for any loss function $L=L(i,d)$, $i=1,\cdots,N$, $d=1,\cdots,D$, and any randomized decision function δ' based on x, y, there is a decision function δ'' based on x alone with $r(\xi,\delta'')\leq r(\xi,\delta')$, where ξ is any probability distribution over $1,\cdots,N$ and $r(\xi,\delta)$ is the average risk using δ against ξ, x is said to be uniformly complete in (x, y). It is shown that x is uniformly complete in (x, y) if and only if x is sufficient for (x, y). *D. Blackwell* (Washington, D. C.).

Nagabhushanam, K. Some aspects of stationary time series. Sankhyā **12**, 109–116 (1952).

1) Let $x(t_1), \cdots, x(t_n)$ be a real stationary process, and

$$E[x(t_i)x(t_j)]=r(t_i, t_j).$$

If $\alpha_1, \cdots, \alpha_n$ are such that $\sum_{j=1}^n \alpha_j r(t_i, t_j)$ = a constant independently of i, then $\sum_{i=1}^n \alpha_i x(t_i)$ is a minimum variance estimator of its expected value. 2) The operation of smoothing carried out on a normal stationary Markoff process destroys the Markoff property. *J. Wolfowitz.*

Kuiper, N. H. Analysis of variance. Statistica, Rijswijk **6**, 149–194 (1952). (Dutch. English summary)

The author shows how terms used in factorial design [R. A. Fisher, The design of experiments, 4th ed., Oliver and Boyd, Edinburgh, 1947] can be conceptionally simplified in the language of linear vector spaces. *H. L. Seal.*

Nair, K. R. Some unsolved problems in experimental designs. Calcutta Statist. Assoc. Bull. **4**, 156–160 (1953).

Mathematical Economics

Kiefer, J. On Wald's complete class theorems. Ann. Math. Statistics **24**, 70–75 (1953).

Conditions are given, weaker than those of Wald [Statistical decision functions, Wiley, New York, 1950; these Rev. 12, 193], but too detailed for summary here, for completeness of (a) the class of minimal strategies and (b) the class of admissible strategies. *D. Blackwell.*

***Arrow, K. J., Barankin, E. W., and Blackwell, D. Admissible points of convex sets.** Contributions to the theory of games, vol. 2, pp. 87–91. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

A point s of a closed convex subset S of k-space is "admissible" if there is no $t\in S$ with $t_i\leq s_i$ for all $i=1,\cdots,k$, $t\neq s$. Let A be the set of admissible points of S and B be the set of those points of A at which there is a supporting hyperplane whose normal has positive components. The authors prove two theorems, of which the first states that A is contained in the closure of B. This is an immediate consequence of a result by Wald [Statistical decision functions, Wiley, New York, 1950, p. 101, Theorem 3.19; these Rev. 12, 193] when the latter is specialized to an Ω (Wald's notation) with finitely many elements. (For removal of boundedness in Wald's result see the paper reviewed above.) The authors then use this result to give another proof of a theorem of Bohnenblust, Karlin, and Shapley [Contributions to the theory of games, Princeton, 1950, pp. 51–72; these Rev. 12, 513]. *J. Wolfowitz* (Ithaca, N. Y.).

***von Neumann, John. A certain zero-sum two-person game equivalent to the optimal assignment problem.** Contributions to the theory of games, vol. 2, pp. 5–12. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

Given n persons, n jobs, and a set of real numbers a_{ij}, each representing the value of the ith person in the jth job, the assignment problem considers the assignments of persons to jobs which yields the maximum total value. The solution is a permutation matrix. A naïve procedure for obtaining the solution would be to test $n!$ permutation matrices. Consider the following two-person, zero-sum, $n^2\times 2n$ game. Move 1: Player I hides in one of the n^2 elements of an $n\times n$ matrix. Move 2: Player II, ignorant of I's choice, tries to find I by guessing either the row or column in which I has hidden. If I is found he pays $a_{ij}>0$ to player II. If I is not found, he pays 0 to player II. Let A be the value of this game. A mixed strategy for I is (x_{ij}), $\sum x_{ij}=1$, where x_{ij} represents his probability of hiding in cell i, j. Theorem: If $\alpha_{ij}=a_{ij}^{-1}$, then the permutations which solve the assignment are related to the extreme optimal strategies for I in the game by $x_{ij}=A\alpha_{ij}^{-1}\delta_{p(i),j}$, where $p(i)$ is the result of permutation p applied to i. The value of the game is the optimal value of the assignment problem. By this means the solution of the assignment problem is reduced to the solution of a

$2n \times n^2$ game, which for large n is potentially a less laborious problem [Brown and von Neumann, same Contributions, vol. 1, Princeton, 1950, pp. 73–79; these Rev. **12**, 514]. Simple transforms also handle the case where a_{ij} may fail to be positive. *S. Sherman* (Sherman Oaks, Calif.).

Gale, David. A theory of n-person games with perfect information. Proc. Nat. Acad. Sci. U. S. A. **39**, 496–501 (1953).

The author proposes a new definition of a solution for non-cooperative n-person games with perfect information based on the theorem: If Γ is the normal form of a game with perfect information, the successive application of the operations D (simultaneously deleting all dominated strategies) and A (simultaneously averaging equivalent strategies with equal weights) will reduce Γ to a game Γ^k with exactly one pure strategy for each player after a finite number of stages. These unique pure strategies for Γ^k are mixed strategies for Γ and are defined to be its "solution". They form an equilibrium point. *H. W. Kuhn.*

***Gale, David, and Stewart, F. M. Infinite games with perfect information.** Contributions to the theory of games, vol. 2, pp. 245–266. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

The notion of a game with perfect information has been defined by von Neumann and Morgenstern [Theory of games and economic behavior, 2nd ed., Princeton, 1947; these Rev. **9**, 50]. The principal result associated with this notion [cf. loc. cit., pp. 123–124] is that finite two-person zero-sum games with perfect information have values which can be achieved by pure strategies. This paper advances the notion of infinite games in extensive form and shows by example that infinite games with perfect information need not be strictly determined. The notion of the topology of a game is introduced as well as the notions: union, intersection, complement, and subgame of a game. These notions involve preliminary definitions not suitable for repetition here. A win-lose game Γ is called absolutely determined if all subgames of Γ and its complement are strictly determined. The connections between the topology of Γ and the property, absolutely determined, are investigated.

S. Sherman (Sherman Oaks, Calif.).

***Blackwell, David. On randomization in statistical games with k terminal actions.** Contributions to the theory of games, vol. 2, pp. 183–187. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

The author shows that in a class of zero-sum two-person games, including statistical multi-decision problems with a fixed sampling plan, any randomized strategy is equivalent to a mixture of a countable number of pure strategies in fixed proportions. *H. W. Kuhn* (Bryn Mawr, Pa.).

***Kuhn, H. W. Extensive games and the problem of information.** Contributions to the theory of games, vol. 2, pp. 193–216. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

The results have already been announced [Proc. Nat. Acad. Sci. U. S. A. **36**, 570–576 (1950); these Rev. **12**, 515].

S. Sherman (Sherman Oaks, Calif.).

***Gillies, D. B., Mayberry, J. P., and von Neumann, J. Two variants of poker.** Contributions to the theory of games, vol. 2, pp. 13–50. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

This is a supplement to "The theory of games . . ." [2nd ed., Princeton, 1947; these Rev. **9**, 50] pp. 186–219 (as promised in a footnote on page 196). It was prepared by D. Gillies (part II) and J. P. Mayberry (part I) from notes of von Neumann, with his advice and collaboration. All good strategies of the game (with bets of two sizes) formulated on pp. 190–191 of the reference are given in part I. Several different cases of classes of good strategies are noted depending on the relationship between the number of possible hands and the ratio of the high bet to the low bet. As the number of hands becomes infinite the good strategies of discrete hand-poker converge to those of the continuous hand-poker in the sense of weak convergence only. In part II, a version of poker, formulated on pp. 209–211 of the reference, is presented. This involves a continuum of hands, does not permit seeing, but does permit of multiplicity of bets, either the closed continuum between positive a and b or a discrete set in the same interval. In the continuous version the expected value for a good strategy is essentially independent of the hand. In the discrete version the impossibility of vanishingly cheap overbids against a given bid leads to a complicated fine structure. The relation between the strategies in the discrete and continuous games are discussed. Recent discussions of poker have appeared by Bellman and Blackwell [Proc. Nat. Acad. Sci. U. S. A. **35**, 600–605 (1949); these Rev. **11**, 192], Kuhn [same Contributions, vol. 1, Princeton, 1950, pp. 97–103; these Rev. **12**, 514], and Nash and Shapley [ibid., pp. 105–116; these Rev. **12**, 514]. *S. Sherman* (Sherman Oaks, Calif.).

***Thompson, G. L. Bridge and signaling.** Contributions to the theory of games, vol. 2, pp. 279–289. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

A simplified model of the play of the last two tricks in bridge is discussed. This is treated as a two-person game or a four-person game depending on whether or not private signals are permitted between partners. *S. Sherman.*

***Thompson, G. L. Signaling strategies in n-person games.** Contributions to the theory of games, vol. 2, pp. 267–277. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

The notion of a signaling strategy is introduced as well as associated behavior strategies and composite strategies in n-person games. It is shown that any strategy attainable by mixed strategies can also be attained by composite strategies. Thus any two-person game can be solved by means of composite strategies. This continues notions considered by Kuhn [Proc. Nat. Acad. Sci. U. S. A. **36**, 570–576 (1950); these Rev. **12**, 515]. Some remarks are made relating signaling and coalitions in n-person games. *S. Sherman.*

***von Stackelberg, Heinrich. The theory of exchange rates under perfect competition.** International Economic Papers, No. 1, pp. 104–159. The Macmillan Co., London-New York, 1951.

Translated from Jahrbücher für Nationalökonomie und Statistik **161**, 1–65 (1949).

Gorman, W. M. Community preference fields. Econometrica **21**, 63–80 (1953).

It is asserted that a given system of personal indifference maps yields a unique community indifference map if and only if the personal Engel curves are parallel straight lines for different individuals at the same prices, in which case the utility possibility map takes the form $\sum u=\text{const.}$ Under these assumptions a dollar increase in total income will always be spent in the same way no matter how it is distributed between persons. However, the existence of Engel curves with identical slopes but different intercepts is very doubtful, except in small intervals of the income range, because quantities consumed have to be non-negative. The author's attempt to deal with this difficulty is not convincing. He assumes that the indifference surfaces do not intersect the coordinate planes, but does not show that this assumption is consistent with non-proportional linear Engel curves. It seems highly unlikely that these assumptions are in fact compatible over the wide ranges of income which would have to be considered in practice.

H. S. Houthakker (Chicago, Ill.).

Rashevsky, N. Outline of a mathematical approach to history. Bull. Math. Biophys. **15**, 197–234 (1953).

Mathematical Biology

Landau, H. G., and Rapoport, A. Contribution to the mathematical theory of contagion and spread of information. I. Spread through a thoroughly mixed population. Bull. Math. Biophys. **15**, 173–183 (1953).

Integro-differential equations are obtained for the number x and the fraction z of infected individuals at time t following the start of an epidemic (or of a rumor) in case the probability $p(t,\tau)$ of transmission depends upon t and also upon the duration τ of the infection in the infected individual making contact. No recovery is assumed. Particular cases studied, however, are $p=p(t)$ and $p=p(\tau)$, with three subcases of the latter: exponential, constant for a finite interval and then vanishing, and vanishing for a finite interval (lag time). On the right of (29) there is an omission of z_0 within the parentheses, and exp before the brackets.

A. S. Householder (Oak Ridge, Tenn.).

Landau, H. G. On dominance relations and the structure of animal societies. II. Some effects of possible social factors. Bull. Math. Biophys. **13**, 245–262 (1951).

[For part I see same Bull. **13**, 1–19 (1951); these Rev. **12**, 843.] The following results are obtained. A uniform bias against reversal of dominance will have no effect on the stationary distribution of the structure of the society. If the probability of dominance is a linear function of the previously established score (number of members dominated), there will be a small tendency for the society to move toward the hierarchy; but this is negligible for large societies. If a member never challenges another whose score exceeds his own by two or more, or if he can never dominate if he should challenge, then the hierarchy is the only stable structure. From the last result it is concluded that social factors which restrict challenges or the probability of dominance could easily account for societies close to the hierarchy, such as are observed in flocks of domestic hens. The effectiveness of social bias in establishing hierarchies is much greater in small societies than in large ones. (From the author's abstract.)

A. S. Householder.

Landau, H. G. On dominance relations and the structure of animal societies. III. The condition for a score structure. Bull. Math. Biophys. **15**, 143–148 (1953).

Two theorems are proved concerning a society with a dominance relation (intransitive, asymmetric). Let v_i $(i=0,\cdots,n-1)$ represent the number of individuals dominated by individual i. Then the sum of any k of these v's is not less than $k(k-1)/2$, and for $k=n$, equality holds. Conversely, given a set of non-negative integers v_i satisfying these relations, then they represent the v's of a possible society. Proof of the converse is by induction and rather long. The second theorem states that if v_i is the largest of the v's, then for any $j\neq i$ either i dominates j, or else there is a k such that i dominates k and k dominates j.

A. S. Householder (Oak Ridge, Tenn.).

Hearon, John Z. Comments on the approximate solution of the diffusion equation. II. Bull. Math. Biophys. **15**, 111–119 (1953).

[For part I see same Bull. **15**, 23–31 (1953); these Rev. **14**, 781.] Rashevsky's approximate solution of the diffusion equation [Mathematical biophysics, rev. ed., Univ. of Chicago Press, 1948] is compared with the exact solution in two special cases: when the rate of production of the diffusing substance is proportional to the concentration, and when it is a specified function of position. In the first case the error is shown to be fairly sensitive to cell size, and to be negligible when the rate is sufficiently small. In the second case a small rate is not sufficient to insure a small error. Equation (5) contains an obvious misprint.

A. S. Householder (Oak Ridge, Tenn.).

De Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. XII. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 255–256 (1953).

Nous étendons la notion de la stationarité et de la stabilité d'une espèce [cf. même Bull. (5) **36**, 683–687 (1950); ces Rev. **12**, 623] aux nouvelles espèces qui dérivent de celle-ci par quantification ou mutation. (Author's abstract.)

A. S. Householder (Oak Ridge, Tenn.).

TOPOLOGY

Neville, E. H. The codifying of tree-structure. Proc. Cambridge Philos. Soc. **49**, 381–385 (1953).

It is shown that a root-tree—that is, a connected linear graph without circuits and with one marked vertex—with n vertices can be completely described (encoded) by a row of $n-2$ symbols each of which may stand for any of the given points. Hence the number of labelled root-trees with n vertices, that is, root-trees in which either all $n-1$ branches, or the $n-1$ vertices other than the root are labelled distinctly (not the number of different trees, as stated by the author), is given by Cayley's formula: n^{n-2}. Three distinct coding procedures are given. *J. Riordan.*

***Kuratowski, Casimir. Topologie. Vol. I.** 3ème ed. Monografie Matematyczne, Tom XX. Polskie Towarzystwo Matematyczne, Warsawa, 1952. xii+450 pp. $7.50.

Except for the correction of a few minor errors, this is identical with the second edition, published in 1948 [these Rev. **10**, 389]. *E. G. Begle* (New Haven, Conn.).

Koutský, Karel. Théorie des lattices topologiques. Publ. Fac. Sci. Univ. Masaryk **1952**, 133–171 (1952). (Czech and Russian summaries)

A topological lattice is not, as one might think, a lattice which is a topological space in which $\cap$ and $\cup$ are continuous operations, but is an abstract lattice in which there is defined a closure operation $x \to \phi(x)$ carrying the lattice into or onto itself. With such objects, one has for a long time studied topology without points [e.g., Nakamura, Proc. Imp. Acad. Tokyo **17**, 5–6 (1941); these Rev. **2**, 342; Monteiro and Ribeiro, Portugaliae Math. **3**, 171–184 (1942); these Rev. **4**, 223]. The author here introduces the study of topology without points and without axioms, considering a perfectly general closure operation ϕ. The article under review gives also a survey of the present status of the theory of topological lattices as well as a well-thought-out discussion of the results of adding one axiom at a time to the requirements imposed upon the closure operator ϕ.

E. Hewitt (Seattle, Wash.).

Myškis, A. D. On the relation of infinitesimal spaces with extensions of topological spaces. Doklady Akad. Nauk SSSR. (N.S.) **84**, 879–882 (1952). (Russian)

A one-to-one correspondence is established between a certain class of proximity ("infinitesimal") structures compatible with a given topological space M and the extensions $R \supset M$ such that R is regular and for every $x \in R-M$ there are $x_n \in M$ with $\lim x_n = x$. *M. Katětov* (Prague).

Papič, Pavle. Sur les espaces admettant une base ramifiée de voisinages. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 30–43 (1953). (Serbo-Croatian. French summary)

Let R be a T_1-space admitting an open basis $\mathfrak{V}$ such that for all $V_1, V_2 \in \mathfrak{V}$, one of the following relations holds: $V_1 \subset V_2$; $V_1 \supset V_2$; $V_1 \cap V_2 = 0$. Such a space is said to have a ramified open basis. A number of properties of these spaces are stated and proved, the following being typical. 1. Every space with a ramified open basis is a 0-dimensional Hausdorff space. 2. Every subspace of a space with a ramified open basis is normal in its relative topology. 3. Every space R with a ramified open basis has an open basis $\mathfrak{D}$ such that for all $x \in R$ and $x \in D \in \mathfrak{D}$, the family of all $E \in \mathfrak{D}$ such that $E \supset D$ is well-ordered by the relation $\supset$. 4. Every space with a ramified open basis and possessing a countable dense subset also has a countable open basis. 5. Every compact space with a ramified open basis is metrizable. The classical space of Baire, consisting of all sequences

$$a = \{a_1, a_2, a_3, \cdots, a_n, \cdots\}$$

of positive integers, with $\rho(a, b) = 1/n$, where $a_1 = b_1, \cdots, a_{n-1} = b_{n-1}, a_n \neq b_n$, is an obvious example of a space with a ramified open basis. [Reviewer's note: the Cartesian product of a finite or countably infinite number of finite T_1-spaces with the Cartesian product topology is another simple example.] Some of the proofs are unnecessarily long.

E. Hewitt (Seattle, Wash.).

Coelho, Renato Pereira. Some properties of regular spaces. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. **2**, 169–183 (1952).

This paper is concerned with topological expansions and contractions, in the terminology of the reviewer [Duke Math. J. **10**, 309–333 (1943); these Rev. **5**, 46], and with semi-regular spaces and regularly open sets, in the terminology of M. H. Stone [Trans. Amer. Math. Soc. **41**, 375–481 (1937)]. A theorem of M. J. Norris [Proc. Amer. Math. Soc. **1**, 754–755 (1950); these Rev. **12**, 434] is also proved. Nearly all of the author's results are contained in the three papers cited above. An interesting new result is the treatment of the family of regularly open subsets of a topological space X (a subset A of X is regularly open if $A^{-\prime-\prime} = A$) as a lattice, with the operations

$$\Delta A_\lambda = \bigcap A_\lambda \quad \text{and} \quad \bigvee A_\lambda = (\bigcup A_\lambda)^{-\prime-\prime}.$$

E. Hewitt (Seattle, Wash.).

Isbell, J. R. Homogeneous spaces. Duke Math. J. **20**, 321–329 (1953).

A space X is bihomogeneous if, for each $x, y \in X$, there is a homeomorphism f of X onto X such that $f(x) = y$ and $f(y) = x$. Moreover, X is involutorially homogeneous if (as above) ff is the identity. The author constructs an example of a space that is bihomogeneous but not involutorially homogeneous thereby solving a problem proposed by van Dantzig. The author gives other results and examples involving various types of homogeneity. He shows that a microhomogeneous linear continuum has the property that each of its points is contained in an open interval which is shrinkable about each of its points. The reviewer presumes that a linear continuum is an ordered set connected (in the topological sense) in the order topology. A space X is microhomogeneous if, for each $x, y \in X$, there are open sets U, V including x, y and a homeomorphism f of U onto V with $f(x) = y$. A space is shrinkable about $x \in X$ if, for each open U including x, there is a homeomorphism of X into U leaving x fixed. Several other results involving linear homogeneous continua are given. Numerous questions are raised; for example: if a space is connected and shrinkable about each of its points, is it locally connected?

A. D. Wallace (New Orleans, La.).

Inagaki, Takeshi. Contribution à la topologie. II. Math. J. Okayama Univ. **2**, 149–184 (1953).

[For part I see same J. **1**, 129–166 (1952); these Rev. **14**, 68.] This is another study attempting to fit countability into topology as part of a more general theory; in this case the larger theory uses a partially ordered set, the character of the space, defined as follows: A topological space R is called quantitative if there is an index set A such that to each point x is assigned a neighborhood system of open sets $v_a(x)$, $a \in A$; define $a \geq_x b$ to mean $v_a(x) \subseteq v_b(x)$; define $a \geq b$ to mean $a \geq_x b$ for every $x \in R$. The ordered system $(A, \geq)$ is called the character of R; if $(A, \geq)$ is directed, the space R is called monotone.

It is proved that $(A, \geq)$ is adequate for the topology of R [J. W. Tukey, Convergence and uniformity in topology, Princeton, 1940; these Rev. **2**, 67]. Notions such as completeness and compactness, usually defined in a metric space in terms of sequences, are defined in R in terms of functions on $(A, \geq)$ into R. This yields, for example, that a complete-in-itself subset of a quantitative Hausdorff space is closed.

To generalize category and Borel sets, let $\mathfrak{a}$ be the cardinal number of A and let α be the smallest corresponding ordinal. R is called exceptional if a covering of A by fewer than $\mathfrak{a}$ sets must include one cofinal subset of $(A, \geq)$. Borel sets and first category are defined using $\leq \mathfrak{a}$ subsets of R; then each set of first category is contained in an F_σ of first category. Defining $E \subseteq R$ to be ordinally compact if every transfinite sequence $\{x_\lambda, \lambda < \beta \leq \alpha\} \subseteq E$ has at least one clus-

ter point in E, in a locally ordinally compact, exceptional quantitative R a set of first category has no interior points.

The rest of the first part of the paper deals with connectedness. The second part of the paper studies uniform spaces as special quantitative spaces; much of this discussion has to do with separation properties of R. *M. M. Day.*

Kaplan, Samuel. Cartesian products of reals. Amer. J. Math. **74**, 936–954 (1952).

The author considers: (I) entire convex topological spaces, i.e., c.t.l. spaces topologically isomorphic to a cartesian product PR_λ of reals; (II) totally fine c.t.l. spaces Y, i.e., Y such that a convex $K\subset Y$ is a nucleus whenever it contains 0 and intersects every one-dimensional subspace in an open interval; (III) "sets of axes" in c.t.l. spaces. Some results: a c.t.l. space is entire if and only if it is complete in its weak topology; a c.t.l. space is entire if and only if it is linearly compact, i.e., if the intersection of a collection of linear varieties is non-empty whenever the collection has the finite intersection property [cf. S. Lefschetz, Algebraic topology, Amer. Math. Soc. Colloq. Publ., v. 27, New York, 1942, p. 78; these Rev. **4**, 84]; a c.t.l. space is totally fine if and only if it is topologically isomorphic with a weak [or "combinatorial", cf. M. Katětov, Acta Fac. Nat. Univ. Carol., Prague no. **181** (1948); these Rev. **10**, 127] product of reals; a totally fine space is complete; an entire space X is reflexive, its conjugate space Y is totally fine, and there is a one-to-one correspondence between the sets of axes in X and the Hamel bases in Y (the correspondence being given by orthogonality). *M. Katětov* (Prague).

Vilenkin, N. Ya. Vector spaces over topological fields. Mat. Sbornik N.S. **32**(74), 195–208 (1953). (Russian)

Let P be a locally bounded topological field, with a distinguished bounded neighborhood V of 0. Let X and Y be respectively the complete direct sum and the weak direct sum of $\aleph$ copies of P. X is awarded the Cartesian product topology. The general neighborhood of 0 in Y is defined as follows: given a set $\{a_i\}$ of elements in P, we take all $\{b_i\}$ in Y satisfying the condition that $\sum c_ib_i \in V$ for any set $\{c_i\}$ with $c_iV\subset a_iV$. The chief purpose of this paper is to characterize abstractly these two spaces. One begins with a topological linear space G over P, and it is assumed outright that the finite-dimensional subspaces of G are homeomorphic to the Cartesian product of copies of P (this is automatically satisfied when P is the field of real numbers). A neighborhood W of 0 is called V-convex if for any x not in W there exists a continuous functional sending W but not x into V. A topology on G is locally V-convex if it has a V-convex base; it is linearly discrete if it is the strongest of all locally V-convex topologies. G is linearly topologized if every neighborhood of 0 contains a closed subspace H such that G/H is linearly discrete. The author proves that a linearly discrete space is (for a suitable $\aleph$) isomorphic to Y, and a linearly compact linearly topologized space is isomorphic to X. The paper concludes with a discussion of locally linearly compact spaces and the appropriate considerations of duality. In a note added in proof it is observed that Kaplan proved essentially the same results in the case where P is the field of real numbers [see the paper reviewed above]. *I. Kaplansky* (Chicago, Ill.).

Gutiérrez Novoa, Lino. On an anti-metric quadratic space. Revista Soc. Cubana Ci. Fis. Mat. **3**, 1–7 (1953). (Spanish).

A set S forms an anti-metric space provided there is defined for certain ordered pairs of elements x, y a real "distance" (x, y) such that (1) $(x, y)>0$, (2) if (x, y) is defined then (y, x) is not, and (3) if (x, y), (y, z) are defined then so is (x, z) and $(x, y)+(y, z)\geqq(x, z)$. Let V' be a vector space over the real field, $\varphi(v)$ a non-definite quadratic form, $v \in V'$, $\varphi(u, v)$ the associated bilinear form, and V the subset of V' with $\varphi(v)=1$. A distance in V is defined by $(u, v)=\cosh^{-1}\varphi(u, v)\geqq 0$. If $v_1 \in V$, denote by V_1 the set of vectors v of V such that $\varphi(v_1, v)\geqq 1$, and let V_2 denote the "opposite" vectors of V_1. The sets V_1, V_2 are metric spaces provided $(u, v)=0$ implies $u=v$. The relation $\varphi(u, v)=1$ $(u, v \in V)$ being an equivalence, the two equivalence classes $\bar{V}_1$, $\bar{V}_2$ are metric spaces, and the transformation $v\rightarrow -v$ is a congruence. Since, if V' is euclidean n-space, $\bar{V}_1$, $\bar{V}_2$ are congruent with hyperbolic $(n-1)$-space, they are generalized hyperbolic spaces. Associated with V is the space $\bar{W}$ generated by the quadratic form $\gamma=-\varphi$. The author shows that $\bar{W}$ is an anti-metric space. *L. M. Blumenthal.*

Cassina, Ugo. Su di un sistema di numerazione a basi variabili. I, II. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15**(**84**), 223–234, 235–240 (1951).

Let $m=[m_1, m_2(x_1), \cdots, m_{k+1}(x_1, \cdots, x_k), \cdots]$ be a sequence of positive integer-valued functions of non-negative integers x_i, $i=1, 2, \cdots$. The sequence m defines a variable base. The author gives a representation of each x, $0\leqq x\leqq 1$, (and eventually for any $x>0$) as a sequence of "digits of the variable base m". Conversely, given such a sequence of "digits", there is a real number between zero and one with these digits as its representation. If all the functions m_k are constant and equal to m_1, then the usual representation to the base m_1+1 is obtained. The main objective of the author is the development of methods to be used in another paper (see part III of the following review). *M. E. Shanks.*

Cassina, Ugo. L'arco nella teoria degli insiemi liberato dal principio della scelta. I, II, III. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15**(**84**), 75–88, 89–101, 102–110 (1951).

In a metric space X, irreducibly connected between two points a and b, there is a natural simple order for the points of X with respect to the point a. The author defines a property which he calls extremal, which, in effect, renders the order relation proper. The author gives in part I a constructive definition of a countable dense subset. Parts II and III are devoted to a proof that X is a simple arc, which the author asserts does not use the axiom of choice. *M. E. Shanks* (Lafayette, Ind.).

Curtis, M. L. Deformation-free continua. Ann. of Math. (2) **57**, 231–247 (1953).

Throughout this review A is a domain of S^n having a continuum M as its frontier. Conditions are sought for A to be ULC^n and M to be LC^n. Examples of J. W. Alexander [Proc. Nat. Acad. Sci. U. S. A. **10**, 8–10 (1924)] and the reviewer [Proc. Nat. Acad. Sci. **34**, 193–196 (1948); these Rev. **10**, 56] shew that neither of the two properties implies the other, and make it likely that a restriction on the situation of M in S^n is needed. Two main types of restriction are considered in this paper. (I) If M is LC^n, a necessary and sufficient condition for A to be ULC^n is that M be "deformation-free" into A by means of a mapping h: $M\times I\rightarrow\bar{A}$ having a "homotopy inverse" g, i.e., a mapping g: $U\rightarrow M\times I$ (where U is a neighbourhood of M in $\bar{A}$) such that (1) $g(U-M)\subseteq M\times(I-0)$, (2) $g(x)=x\times 0$ if $x \in M$, (3) $hg\simeq 1$ in $h(M\times I)$ by a homotopy leaving all points of M fixed. (II) a sequence of closed sets, M_i, in S^n converging to M

converges "homotopy-p-regularly" if, given $\epsilon>0$, there exist i_0 and $\delta>0$ such that, when $q\leq p$ and $i>i_0$, every map f: $S^q\to M_i$ of diameter $<\delta$ is null-homotopic in a set of diameter $<\epsilon$ in M_i. The sequence of closed sets M_i in A is "locally separating, rel. M", if it converges to M and, given $x\in M$, each point of A in some ϵ-neighbourhood $S(x,\epsilon)$ of x is separated from M in $S(x,\epsilon)$ by M_i for all sufficiently large i. Theorem 4.4. If the M_i are closed LC^{p-1} subsets of A and are both homotopy-p-regularly convergent to M, and locally separating rel. M, then A is ULC^p and M is LC^p.

A number of other results are established en route, and examples are given of a 3-gcm not 1-lc over the integers, and a 3-gcm which, though LC^3, is not a 3-manifold.

M. H. A. Newman (Manchester).

White, Paul A. Regular convergence of manifolds with boundary. Proc. Amer. Math. Soc. 4, 482–485 (1953).

Earlier results of the author and of G. T. Whyburn regarding the "regular" convergence of certain types of configurations are extended to the case of orientable n-dimensional generalized manifolds (n-gm's) with boundary. The convergence is of the r-regular type, embodying a uniformity condition strong enough to force certain local-connectedness properties of the elements of a sequence to hold for the limit set of the sequence. The main result is that if $\{M_i\}$ is a sequence of orientable n-gm's with boundaries $\{K_i\}$, such that $\{M_i\}$ approaches a set M $(n-1)$-regularly and $\{K_i\}$ approaches a set K $(n-2)$-regularly, then M is an orientable n-gm with boundary K. *R. L. Wilder.*

Homma, Tatsuo, and Terasaka, Hidetaka. On the structure of the plane translation of Brouwer. Proc. Japan Acad. 29, 13–16 (1953).

The results announced here without proof are said to give the complete structure of generalized translations, that is, of sense-preserving fixed-point-free homeomorphisms of the Euclidean plane E^2 onto itself. Let f be a generalized translation. A point p of E^2 is irregular if, for any neighborhood U of p, the sets $U, fU, f^2U, \cdots$ have a nonempty cluster set. Let O denote the open set which consists of the totality of regular (i.e., non-irregular) points. A regular streamline of f is a closed set which is the topological image of a line and is mapped onto itself by f. The structure theorem asserts that E^2 is the union of disjoint sets $O_1, O_2, \cdots; O_1', O_2', \cdots;$ F. The O's and O''s are simply connected open sets of regular points, the O's being components of O. For each n, $fO_n=O_n$ and $fO_n'\cap O_n'=\phi$. Each O_n can be filled with a regular family of regular streamlines. F is a closed set of irregular points and is the union of a system of sets called singularity blocks which, unlike the streamlines, are uniquely determined by f. Generalized translations can be constructed with preassigned sets, satisfying certain conditions, as singularity blocks. The definition of singularity block depends on the concept of translation field with which Brouwer's difficult "translation theorem" is concerned. Whether the present results make use of the translation theorem or, on the contrary, imply that theorem will no doubt be clear when the full exposition is available. *P. A. Smith.*

Dowker, Yael Naim. The mean and transitive points of homeomorphisms. Ann. of Math. (2) 58, 123–133 (1953).

Given a homeomorphism of a compact metric space Ω onto itself, a point p is called mean, i.e., quasi-regular ($p\in Q$) if the average of any continuous function over the positive semi-orbit of p exists, and p is called transitive ($p\in Q_T$) if this mean value is expressible as an integral with respect to an ergodic Borel measure. It was shown by Kryloff and Bogoliouboff [Ann. of Math. (2) 38, 65–113 (1937)] that the sets Q and Q_T are non-empty and of maximal invariant measure. In this paper the cardinal number and category of Q and Q_T are determined under various conditions. The results: (1) both sets are infinite whenever Ω is infinite; (2) both are of power c when Ω is infinite and locally connected; and (3) both are of first category if the orbit of some point not in Q is dense. An example, where Ω is a subset of 3-space, is constructed to show that both Q and Q_T may be denumerable even though Ω has power c. The proofs are based on results of Kryloff and Bogoliouboff and on some theorems due to Kerekjarto and Montgomery, suitably modified. *J. C. Oxtoby.*

Noguchi, Hiroshi. On mappings defined on 2-spheres. Kōdai Math. Sem. Rep. 1952, 109–110 (1952).

It is shown here that if S is a 2-sphere, with center 0, if θ is an angle, $0<\theta<\pi$, and if f is a mapping of S into the plane, then there exist two points, P and Q, on S such that the vectors OP and OQ make an angle θ and $f(P)=f(Q)$. This result is shown to remain correct if the plane is replaced by any 2-dimensional manifold with genus not zero.

E. G. Begle (New Haven, Conn.).

Bott, R. On the third symmetric potency of S_1. Fund. Math. 39 (1952), 264–268 (1953).

Borsuk [Fund. Math. 36, 236–244 (1949); these Rev. 12, 42] investigated the third symmetric product $S_1^{(3)}$ of the circumference S_1 and concluded that $S_1^{(3)}$ is the cartesian product of a circumference and a 2-sphere. An error in the last step of Borsuk's proof is pointed out and corrected in the present note. The correct result is that $S_1^{(3)}$ is a 3-sphere.

E. G. Begle (New Haven, Conn.).

Ganea, Tudor. Remark on R-equivalent spaces. Acta Math. Acad. Sci. Hungar. 3 (1952), 295–297 (1953). (Russian summary)

According to Borsuk, two spaces A and B are R-equivalent if there are homeomorphisms f: $B\to A_1\subset A$ and g: $A\to B_1\subset B$ such that A_1 is a retract of A and B_1 a retract of B. It is shown here that if $\varphi=fg$ has equicontinuous positive powers, then A and B have the same homotopy type. This is a partial answer to a question raised by Borsuk.

E. G. Begle (New Haven, Conn.).

Kapuano, Isaac. Sur les courbes dont l'homéomorphie avec une circonférence se prolonge à R^3. C. R. Acad. Sci. Paris 236, 1845–1847 (1953).

The author states and indicates a proof for the following. Suppose that C, C' are subsets in R^3 such that there are homeomorphisms h, h' of R^3 onto itself with $h(C)$, $h'(C')$ circles. Assume also that C and C' are homotopic in the open domain D. Then there is a homeomorphic deformation d of C onto C' leaving fixed the points of R^3-D. If C is null-homotopic in R^3-C', then d may be taken to leave fixed the points of C. *A. D. Wallace* (New Orleans, La.).

Moise, Edwin E. Affine structures in 3-manifolds. VI. Compact spaces covered by two Euclidean neighborhoods. Ann. of Math. (2) 58, 107 (1953).

[For part V see Ann. of Math. (2) 56, 96–114 (1952); these Rev. 14, 72.] It is shown that if M is a compact metric space which is the sum of two open sets U and V, each homeomorphic to Euclidean 3-space, then M is a topological 3-sphere. *S. S. Cairns* (Urbana, Ill.).

Dedecker, Paul. Quelques notions relatives aux structures locales. C. R. Acad. Sci. Paris **236**, 771–774 (1953).

The author studies the general theory of local structures, initiated by Ehresmann [same C. R. **234**, 587–589 (1952); these Rev. **13**, 780]; a number of new concepts are introduced, e.g., homologous local structures, isomery, compatibility of local structures, mappings of spaces with local structures. A general definition of germ of a given kind at a point in a space with local structure is given; this generalizes the germs considered by Ehresmann [lectures, Rio de Janeiro, 1952] and the germs of analytic subvarieties of analytic manifolds considered by H. Cartan [Séminaire de l'Ecole Normale Supérieure, Paris, 1951–52]. On the totality of germs at all points a local structure can be defined which is universal in the sense that any map into the original space, compatible with the structures, can be lifted to this universal structure. *H. Samelson* (Princeton, N. J.).

Hirsch, Guy. Sur les invariants attachés aux sections dans les espaces fibrés. Colloque de Topologie de Strasbourg, 1951, no. VII, 13 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1952.

La situation étudiée est la suivante (nous ne conservons pas les notations de l'auteur): E est un espace fibré de base B, de fibre F; l'homologie $H_*(E)$ (resp. $H_*(B)$, $H_*(F)$) et la cohomologie $H^*(E)$ (resp. $H^*(B)$, $H^*(F)$) sont prises avec coefficients dans un corps. On suppose que le système local formé par l'homologie des fibres est simple sur la base B, et que $H_*(F)\to H_*(E)$ est biunivoque (fibre "totalement non homologue à zéro"). Il est probable que l'on suppose aussi $H^*(B)$ de dimension finie. Alors (Leray) il existe un isomorphisme d'espaces vectoriels

$$(1)\qquad H^*(B)\otimes H^*(F)\approx H^*(E)$$

compatible avec les structures de module sur $H^*(B)$; un tel isomorphisme n'est pas unique, mais est déterminé par la donnée d'une application linéaire $f: H^*(F)\to H^*(E)$ telle que la composée de f et de l'application naturelle $H^*(E)\to H^*(F)$ soit l'application identique de $H^*(F)$. L'auteur suppose donnée une section s: $B\to E$, tout au moins au-dessus du squelette d'une dimension convenable; d'où un homomorphisme s^*: $H^*(E)\to H^*(B)$. On astreint alors f à avoir son image dans le noyau de s^*. Si, par l'isomorphisme (1) défini par f, on transporte à $H^*(B)\otimes H^*(F)$ la multiplication de $H^*(E)$, on voit que celle-ci est déterminée par la multiplication des éléments de $H^*(F)$, c'est-à-dire par une application $g: H^*(F)\otimes H^*(F)\to H^*(B)\otimes H^*(F)$. Cela posé, il semble que l'auteur profite de l'indétermination de f pour réduire l'application g à une forme plus ou moins canonique, afin d'obtenir des invariants de l'espace fibré. Introduisant une deuxième section s': $B\to E$, il définit un grand nombre d'"invariants" attachés au couple s, s', et prétend retrouver ainsi notamment la deuxième obstruction définie par Hopf. *H. Cartan* (Paris).

Thom, René. Espaces fibrés en sphères et carrés de Steenrod. Ann. Sci. Ecole Norm. Sup. (3) **69**, 109–182 (1952).

Throughout this paper, the author uses the term "fibre space" to mean a space which is locally a product space. To be precise, E is a fibre space over B with fibre F with respect to the "projection" p: $E\to B$ if every point $a\in B$ has a neighborhood U such that there exists a homeomorphism h of $U\times F$ onto $p^{-1}(U)$ having the property that $p[h(x, y)]=x$ for any $x\in U$ and $y\in F$. Unless explicit mention is made to the contrary, no assumption is made about the existence of a structural group. It will be most convenient to consider this paper by chapters.

Chapter I is concerned with the general theory of fibre spaces E for which the fibre F is a $(k-1)$-dimensional sphere and the base B is any locally compact space. To any such fibre space are associated two other fibre spaces, A and A', having the same base B, and as fibre a closed k-cell and an open k-cell, respectively. A is the mapping cylinder of the projection p, E is a subspace of A, and $A'=A-E$. The cohomology groups $H^r(B)$ and $H^r(A)$ are obviously isomorphic; this isomorphism is denoted by $j: H^r(B)\to H^r(A)$. The author proves that there exists a natural isomorphism ϕ^* of $H^r(B)$ onto $H^{r+k}(A')$ for all dimensions r. Consider the exact sequence defined by the injection of the closed subspace E into A:

$$\cdots\to H^{r-1}(E)\to H^r(A')\to H^r(A)\to H^r(E)\to\cdots.$$

By using the isomorphisms ϕ^* and j, one obtains another exact sequence, the so-called Gysin sequence:

$$\cdots\to H^{r-1}(E)\to H^{r-k}(B)\to H^r(B)\to H^r(E)\to\cdots.$$

The author indicates various other properties of the isomorphisms ϕ^* and j, including their behavior with respect to cup products.

In chapter II the Steenrod squares are applied to the study of fibre spaces. By means of the isomorphism ϕ^*, the Steenrod squares in A', Sq^i: $H^q(A')\to H^{q+i}(A')$, define homomorphisms θ^i: $H^r(B)\to H^{r+i}(B)$ in B. Let $\omega\in H^0(B)$ denote the unit of the cohomology ring of B. The author defines "generalized characteristic classes", W^i, by the formula $W^i=\theta^i(\omega)$. He then proves the following fundamental theorem: In the case of a sphere bundle (with the rotation group as structural group) the generalized characteristic classes are the same as the Stiefel-Whitney classes defined in the classical manner. Many known properties of the Stiefel-Whitney classes follow from this theorem and the properties of the Steenrod squares.

The third chapter is concerned principally with the study of the imbedding of one manifold in another. The author first proves a theorem to the effect that if Y is a closed subset of the topological space E, and Y and E satisfy certain "smoothness" conditions, then the q-dimensional singular homology group, $H_q(Y)$, is naturally isomorphic to the inverse limit of the singular homology groups $H_p(U)$ as U ranges over all open neighborhoods of Y in E. In particular, the necessary smoothness conditions are always satisfied in case Y and E are separable manifolds (without any hypothesis of differentiability). Now let V be a p-dimensional manifold which is a closed subset of an n-dimensional manifold M, and let $H_r(\mathcal{V}_V{}^M)$ denote the inverse limit of the homology groups $H_r(U)$ as U ranges over all open neighborhoods of V; similarly, $H^r(\mathcal{V}_V{}^M)$ will denote the inverse limit of the cohomology groups $H^r(U)$. By the Poincaré duality theorem for open manifolds, $H_r(U)\approx H^{n-r}(U)$; it follows that $H_r(\mathcal{V}_V{}^M)\approx H^{n-r}(\mathcal{V}_V{}^M)$. By the discussion above, $H_r(V)\approx H_r(\mathcal{V}_V{}^M)$. By the Poincaré duality theorem again, $H_r(V)\approx H^{p-r}(V)$. Composition of these three isomorphisms leads to an isomorphism ψ: $H^{p-r}(V)\to H^{n-r}(\mathcal{V}_V{}^M)$. The Steenrod squares, Sq^i: $H^q(U)\to H^{q+i}(U)$, enable one to define squaring operations Sq^i: $H^q(\mathcal{V}_V{}^M)\to H^{q+i}(\mathcal{V}_V{}^M)$ in the inverse limit groups. The author now defines "generalized normal classes", $\bar{W}^i\in H^i(V)$, by the condition

$$\psi(\bar{W}^i)=\mathrm{Sq}^i[\psi(\omega)],$$

where $\omega\in H^0(V)$ is the unit class. He then proves the fundamental theorem that if V and M are differentiable manifolds,

and V is imbedded differentiably in M, then the generalized normal classes are precisely the Stiefel-Whitney classes of the bundle of unit normal vectors. As a corollary, it follows that the Stiefel-Whitney classes of the normal bundle are independent of the differential structure on M and V. Next, it is proved that the Stiefel-Whitney classes of the bundle of unit tangent vectors to a differentiable manifold are also independent of the differential structure. This follows from consideration of the imbedding of the manifold V in the product $V\times V$ by means of the diagonal map, and then observing that for this imbedding, the bundles of tangent vectors and normal vectors are isomorphic. The above mentioned corollary can be applied to complete the proof.

The author shows that the condition that V be a manifold can be relaxed to some extent in the above discussion. Results are obtained on the imbedding of a locally contractible space in an n-manifold. In particular, necessary conditions are obtained for the imbeddability of a finite-dimensional compact metric A.N.R. in Euclidean m-space.

The fourth chapter is devoted to several theorems about the invariance of the tangent bundle of a differentiable manifold. These theorems prove the invariance of the Stiefel-Whitney classes by a completely different method. The general concept of "fibred homotopy type" for fibre spaces having a given base space and fibre is introduced. It is proved that the fibred homotopy type of the bundle of tangent vectors to a differentiable manifold is independent of the differentiable structure. Also, the Stiefel-Whitney classes of fibre spaces having spheres for fibres are invariants of the fibred homotopy type.

The fifth chapter is concerned with manifolds with boundary. The author's results in this chapter were announced previously [Colloque de Topologie de Strasbourg, 1951, no. V, Bibliothèque Nation. et Univ. de Strasbourg, 1952; these Rev. 14, 492]. *W. S. Massey* (Providence, R. I.).

Thom, René. Sous-variétés et classes d'homologie des variétés différentiables. I. Le théorème général. C. R. Acad. Sci. Paris **236**, 453–454 (1953).

Thom, René. Sous-variétés et classes d'homologie des variétés différentiables. II. Résultats et applications. C. R. Acad. Sci. Paris **236**, 573–575 (1953).

Let W^p be a p-dimensional sub-manifold which is imbedded differentially in the compact, n-dimensional, differentiable manifold, V^n. The inclusion map $i: W^p\rightarrow V^n$ induces a homomorphism i_* of the homology group $H_p(W^p)$ into $H_p(V^n)$; let $z\in H_p(V^n)$ denote the image under i_* of the fundamental homology class of W^p; the class z is then said to be "realizable" by the sub-manifold W^p. In these two notes the author considers the problem of determining necessary and sufficient conditions in order that a given homology class $z\in H_p(V^n)$ shall be realizable by a sub-manifold.

First of all, he states a general theorem giving a necessary and sufficient condition that a homology class $z\in H_p(V^n)$ shall be realizable by a sub-manifold W^p in such a way that the bundle of unit normal vectors of W^p admits a preassigned subgroup G of the full orthogonal group as structural group. He then applies this general theorem to the following three special cases: G is the trivial group, having only one element; G is the full orthogonal group; G is the group of all proper rotations. Among the results obtained are the following. (a) If V^n is orientable, then for every integral homology class $z\in H_{n-k}(V^n)$ with k odd or $n<2k$ there exists a non-zero integer N depending only on k and n such that the homology class $N\cdot z$ is realizable by a sub-manifold W^{n-k} whose normal bundle is a product bundle. (b) Using integers mod 2 for coefficients, all the homology classes of the following groups are realizable by sub-manifolds: $H_{n-1}(V^n)$ for all n; $H_{n-2}(V^n)$ for $n<6$; $H_i(V^n)$ for $i<[n/2]$. (c) In a compact orientable manifold V^n, all the following integral homology classes are realizable by orientable sub-manifolds: $H_{n-1}(V^n)$ and $H_{n-2}(V^n)$ for all n; $H_i(V^n)$ for $i\leqq 5$; $H_i(V^n)$ for $i\leqq n<9$. (d) If V^n is an orientable manifold and k is a positive even integer, then for every homology class $z\in H_{n-k}(V^n)$ there exists a non-zero integer N such that the homology class $N\cdot z$ is realizable by a sub-manifold. (e) There exists a compact 11-dimensional manifold V^{11} and a modulo 2 homology class of $H_8(V^{11})$ which is not realizable by a sub-manifold; similarly, there exists a V^{17} and an integral homology class of $H_{14}(V^{17})$ which is not realizable.

W. S. Massey (Providence, R. I.).

Thom, René. Sur un problème de Steenrod. C. R. Acad. Sci. Paris **236**, 1128–1130 (1953).

N. E. Steenrod has proposed the following problem: Given a finite polyhedron K and a homology class $z\in H_r(K)$, does there exist an r-dimensional manifold W and a continuous map f of W into K such that z is the image of the fundamental homology class of W under the induced homomorphism f_*? In the note reviewed above (part II) the author proved that if one uses integers modulo 2 for coefficients, the answer is affirmative; in fact, one may even assume that W is compact and differentiable. The method used is as follows: Imbed K rectilinearly in Euclidean space R^m of some dimension m, and choose a closed neighborhood Q of K in R^m which is a manifold with boundary and is "small" enough so that K is a retract of Q. Let V^m denote the manifold obtained by "doubling" Q. It is readily seen that the cohomology and homology groups of V^m contain those of K as subgroups. If the homology class z, considered as a homology class in V^m, can be realized by a submanifold of V^m, then it is readily seen that the answer to Steenrod's problem is affirmative.

In the present note, the author proves a sort of converse theorem: if the answer to Steenrod's problem is affirmative for the class $z\in H_r(K)$ then one can imbed K in R^m (for sufficiently large m) in such a way that z, considered as a homology class in the V^m constructed by the above process, can be realized by a submanifold of V^m. This converse theorem enables him to apply the results of his previous note [loc. cit.] to the problem of Steenrod. The main results are the following: (a) Every integral r-dimensional homology class of K for $r\leqq 6$ is the image of the fundamental class of a compact differential manifold; (b) for every dimension $r\geqq 7$, there exist integral homology classes of finite polyhedra which are not the image of the fundamental class of a compact differentiable manifold.

In the derivation of these results, the author also uses certain techniques introduced in his thesis [see p. 152 of the paper reviewed second above], modified so as to involve Steenrod's reduced pth powers for p an odd prime.

W. S. Massey (Providence, R. I.).

Steenrod, N. E. Homology groups of symmetric groups and reduced power operations. Proc. Nat. Acad. Sci. U. S. A. **39**, 213–217 (1953).

L'auteur généralise sa définition antérieure des "puissances réduites" [Ann. of Math. (2) **56**, 47–67 (1952); ces Rev. **13**, 966]. Soient n un entier, π un sous-groupe du

groupe symétrique S_n de degré n; si X est un complexe cellulaire, $u \in H^q(X; B)$ une classe de cohomologie à coefficients dans B (groupe abélien), $c \in H_i(\pi; A)$ une classe d'homologie de π à coefficients dans A (groupe abélien dans lequel opère π), on a une puissance réduite $u^n/c \in H^{nq-i}(X; A \otimes_\pi B^n)$, où B^n désigne le produit tensoriel $B \otimes \cdots \otimes B$ (n facteurs) où S_n opère de manière évidente si q pair (resp. en multipliant par la signature de la permutation si q impair), et où $A \otimes_\pi B^n$ désigne le quotient de $A \otimes B^n$ par le sous-groupe engendré par les éléments de la forme

$$a \otimes e - \sigma a \otimes \sigma e \quad (a \in A, e \in B^n, \sigma \in \pi).$$

Principales propriétés: $u^n/c=0$ si $nq-i<q$; si $f: X \to Y$ est une application cellulaire, on a

$$(1) \qquad f^*(u^n/c) = (f^*u)^n/c;$$

si $\pi \subset \pi' \subset S_n$, et si A est un π'-module, soient λ_*: $H_i(\pi; A) \to H_i(\pi'; A)$, $\lambda': A \otimes_\pi B^n \to A \otimes_{\pi'} B^n$; alors

$$(2) \qquad u^n/(\lambda_* c) = \lambda'(u^n/c);$$

si $\partial_*: H_i(\pi; Z_m) \to H_{i-1}(\pi; Z)$ et $\delta^*: H^q(X; Z_m) \to H^{q+1}(X; Z)$ désignent les homomorphismes de Bockstein, on a

$$(3) \qquad u^n/\partial_* c = (-1)^{i+1}\delta^*(u^n/c)$$

$$\text{pour} \qquad u \in H^q(X; Z), \quad c \in H_i(\pi; Z_m).$$

Tous les résultats valent en cohomologie relative.

H. Cartan (Paris).

Steenrod, N. E. Cyclic reduced powers of cohomology classes. Proc. Nat. Acad. Sci. U. S. A. **39**, 217–223 (1953).

Cette note fait suite à une autre analysée ci-dessus dont nous conservons les notations. On prend pour π un sous-groupe cyclique d'ordre n du groupe symétrique S_n, et pour A l'un des groupes Z et Z_n (où π opère trivialement). La formule (3) du compte rendu précédent détermine u^n/c, $c \in H_{2i-1}(\pi; Z)$ ou $H_{2i-1}(\pi; Z_n)$, quand on connaît u^n/e_{2i}, e_{2i} élément canonique de $H_{2i}(\pi; Z_n)$. Si n est premier impair, $u^n/e_{2i}=0$ si $(nq-2i)-q \not\equiv 0$ $(2n-2)$ (théorème de Thom): cela résulte de la formule (2) du compte rendu précédent, et du fait qui h_{2i}: $H_{2i}(\pi; Z_n) \to H_{2i}(S_n; Z_n)$ est nul pour $2i \not\equiv 0$ $(2n-2)$ si S_n opère trivialement sur Z_n, resp. nul pour $2i+n-1 \not\equiv 0$ $(2n-2)$ si $\sigma \in S_n$ opère sur Z_n par la signature de σ (la démonstration repose sur les propriétés des racines primitives de l'entier premier n).

Les puissances réduites de $u \otimes v$ s'expriment comme sommes de produits (tensoriels) de puissances réduites de u et de v (utiliser la multiplication des cochaînes du groupe π). Pour n impair, $u^n/e_{q(n-1)}$ est proportionnel à u; et, si Y est un sous-complexe de X, et qu'on note $\delta: H^r(Y) \to H^{r+1}(X, Y)$, $(\delta u)^n/e_{2i+n-1}$ est proportionnel à $\delta(u^n/e_{2i})$. Pour n premier impair, on peut faire disparaître ces coefficients encombrants: suivant Serre, on introduit des facteurs non nuls $c(n, q) \in Z_n$ de manière que les opérations $\mathcal{P}^s$:

$$H^q(X; Z_n) \to H^{q+2s(n-1)}(X; Z_n)$$

définies par $\mathcal{P}^s u = c(n, q) u^n/e_{2i}$ $(i=(q-2s)(n-1))$ jouissent des propriétés suivantes: $\mathcal{P}^0=$identité, $\mathcal{P}^s=0$ si $s>q/2$, $\mathcal{P}^{q/2}u=u^n$ si q pair,

$$\mathcal{P}^s(u \otimes v) = \sum_{0 \le t \le s} \mathcal{P}^t u \otimes \mathcal{P}^{s-t} v, \quad \mathcal{P}^s f^* = f^* \mathcal{P}^s, \quad \mathcal{P}^s \delta = \delta \mathcal{P}^s.$$

La dernière relation vaut si δ désigne un homomorphisme de suspension.

H. Cartan (Paris).

Dolbeault, Pierre. Sur la cohomologie des variétés analytiques complexes. II. C. R. Acad. Sci. Paris **236**, 2203–2205 (1953).

The notation is the same as that of the author's earlier note of the same title [same C. R. **236**, 175–177 (1953); these Rev. **14**, 673]. The following three results are established:

1) For all integers p, $q \geqq 0$ we have the commutative diagram

$$\begin{array}{ccccc} \cdots \to & K^{p+1,q}(V) & \xrightarrow{f'} & K^{p,q+1}(V) & \xrightarrow{k'} \\ & \updownarrow & & \updownarrow & \\ \cdots \to & H^q(V, \Phi^{p+1}) & \to & H^{q+1}(V, \Phi^p) & \to \end{array}$$

$$\begin{array}{cccc} H^{p,q+1}(V) & \xrightarrow{h} & K^{p+1,q+1}(V) & \to \cdots \\ \updownarrow & & \updownarrow & \\ H^{q+1}(V, \Omega^p) & \to & H^{q+1}(V, \Phi^{p+1}) & \to \cdots \end{array}$$

where the rows are exact and the vertical homomorphisms are those established in the author's earlier note. The homomorphisms f', k', h of the first row are induced respectively by injection, projection, and by $(-1)^{q+1}\bar{\partial}$. The homomorphisms of the second row are those defined by the exact sequence $0 \to \Phi^p \to \Omega^p \to \Phi^{p+1} \to 0$.

2) Denote by X a principal fibre space with base V and fibre a complex abelian Lie group $G \approx C^n/\Pi$, where Π is a discrete subgroup of complex n-space C^n. Let $\Omega_n{}^p$ be the stack (faisceau) of germs of holomorphic differential forms of degree p with coefficients in C^n, and let $\Phi_n{}^p$ be the substack of $\Omega_n{}^p$ composed of those germs which are closed under $d=\partial+\bar{\partial}$. If $\mathcal{G}$ is the stack of germs of analytic maps of V into $\mathcal{G}$, we have the homomorphism λ': $\mathcal{G} \to \Phi_n{}^1$ which maps $\tau \in \mathcal{G}_x$ into $d\tau \in (\Phi_n{}^1)_x$. An element $u \in H^1(V, \mathcal{G})$ determines a class of analytically equivalent fibre spaces X, and a fibre space X is said to have an invariant analytic connection if the image of λu in the homomorphism $H^1(V, \Phi_n{}^1) \to H^1(V, \Omega_n{}^1)$ is zero. In order that an element of $H^2(V, C_n)$ should be the topological obstruction of a fibre space X possessing an invariant analytic connection, it is necessary and sufficient that it be represented by a differential form of type $(2, 0)$. If V is compact Kähler, this result has been given by Blanchard [ibid. **234**, 284–286 (1952); these Rev. **14**, 903].

3) Let Γ be the fundamental group of V, $\tilde{V}$ its universal covering, q: $\tilde{V} \to V$. Denote by q^* the mapping defined by q of the differential forms of V into those of $\tilde{V}$, by ξ^* the mapping of the differential forms of $\tilde{V}$ defined by an automorphism $\xi \in \Gamma$ of $\tilde{V}$. A closed meromorphic p-form A on $\tilde{V}$ is said to be additive if, for every $\xi \in \Gamma$, $A-\xi^*A$ is the image by q^* of a closed holomorphic p-form on V. The form A defines on V a system of singular parts, that is, an element of $H^0(V, m^p/\Phi^p)$ where m^p is the stack of germs of closed meromorphic p-forms. Given $s \in H^0(V, m^p/\Phi^p)$, there exists on $\tilde{V}$ an additive p-form A having s as system of singular parts if the image of s in $K^{p,1}(V)$ is represented by a sum of products of closed holomorphic p-forms and of closed 1-forms. The condition is also necessary if the space of closed holomorphic p-forms on V has finite dimension or if the first Betti group of V has finite dimension. This result, applied to 1-forms on a compact Kähler variety V, gives a theorem of Kodaira [de Rham and Kodaira, Harmonic integrals, Institute for Advanced Study, Princeton, 1950; these Rev. **12**, 279].

D. C. Spencer (Princeton, N. J.).

GEOMETRY

Thébault, Victor. Sur des cercles et des sphères particuliers de Tücker. Mathesis 62, 111–119 (1953).

Raljević, Šefkija. Sur une généralisation de la droite et du segment d'Euler. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 8, 47–48 (1953). (Serbo-Croatian. French summary)

van der Vaart, H. R. The content of certain spherical polyhedra for any number of dimensions. Experientia 9, 88–90 (1953).

An $(n-1)$-dimensional spherical simplex $P_1P_2\cdots P_n$ is called an orthoscheme if the $n-1$ edges $P_1P_2, P_2P_3, \cdots, P_{n-1}P_n$ are all orthogonal to one another, in which case the shape of the simplex is determined by the $n-1$ dihedral angles opposite to these edges (since all the remaining dihedral angles are right angles). Schläfli [Gesammelte mathematische Abhandlungen, Bd. 2, Birkhäuser, Basel, 1953, p. 250; these Rev. 14, 833] considered the $(n-1)$-dimensional content of the orthoscheme as a function of these $n-1$ angles. The author has obtained a new expression for this function as a finite series whose terms are multiple integrals of algebraic functions. It may be significant that the terms of his series correspond to those of a series used by the reviewer [Duke Math. J. 18, 765–782 (1951), p. 768 (2.1); these Rev. 13, 528] in connection with the product of the reflections in the faces of the simplex.

H. S. M. Coxeter (Toronto, Ont.).

Bilinski, Stanko. Homogeneous nets on closed orientable surfaces. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke 277, 129–164 (1950). (Serbo-Croatian)

Let $\{n, k\}$ denote a regular map of n-gons, k at each vertex, covering a closed surface of genus p. Let $\alpha_0, \alpha_1, \alpha_2$ denote the numbers of vertices, edges, regions. Solving the equations

$$k\alpha_0 = 2\alpha_1 = n\alpha_2, \quad \alpha_0 - \alpha_1 + \alpha_2 = 2(1-p),$$

the author finds that

$$\alpha_0 = 2nt, \quad \alpha_1 = knt, \quad \alpha_2 = 2kt,$$

where $t = 2(p-1)/(kn-2k-2n)$ [except when $p=1$, in which case $t=(b^2+c^2)/8$ for $\{4,4\}_{b,c}$, $t=(b^2+bc+c^2)/6$ for $\{3,6\}_{b,c}$ and $\{6,3\}_{b,c}$; see Coxeter, Bull. Amer. Math. Soc. 56, 413–455 (1950), p. 421; these Rev. 12, 350]. He proves that the existence of an $\{n, k\}$ of genus $p>1$ implies that of an $\{n, k\}$ of genus $\nu(p-1)+1$ for every positive integer ν. He conjectures that such maps exist whenever the values of n, k, and p (>1) are such as to give the α's integral values. In support of this conjecture, he tabulates the 25 possible solutions when $p=2$, and finds that all of them can be realized topologically. It follows that these 25 $\{n, k\}$'s exist for every $p \geq 2$.

Most of these results were anticipated by W. Threlfall [Abh. Math.-Phys. Kl. Sächs. Akad. Wiss. 41, no. 6 (1932), pp. 44–45], who observed that the existence of such a map depends on the existence of a group of order $2knt$ generated by operations S and T which satisfy (among other relations)

$$S^n = T^2 = (ST)^k = 1.$$

However, the author considers the more general problem of semi-regular maps $R[n_1, n_2, \cdots, n_r]$ where the regions at each vertex consist of various polygons: an n_1-gon, an n_2-gon, and so on, in the given cyclic order [Bilinski, same journal 271, 145–255 (1948); these Rev. 11, 197]. In particular, the existence of a regular map $\{n, k\} = R[n, n, \cdots, n]$ implies that of various semi-regular maps analogous to the Archimedean solids. In the notation of Coxeter [Math. Z. 46, 380–407 (1940), p. 394; these Rev. 2, 10], these maps are:

$$\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\} = R[n, k, n, k], \qquad t\{n, k\} = R[2n, 2n, k],$$
$$r\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\} = R[4, n, 4, k], \qquad t\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\} = R[4, 2n, 2k],$$
$$s\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\} = R[3, 3, n, 3, k].$$

But they by no means exhaust the possibilities for semi-regular maps; e.g., the author's first two figures show an $R[3, 4, 6, 4]$ of genus 1 which is not $r\left\{\begin{smallmatrix} 3 \\ 6 \end{smallmatrix}\right\}_{4,6}$ although it has the same numbers of vertices, edges, and regions.

The rest of his sixteen figures, many of which are very beautiful, show maps $\{3, 7\}$, $\{3, 9\}$, $\{7, 3\}$, $\{5, 4\}$, $\{6, 6\}$, $\{18, 3\}$ of genus 2, $\{4, 6\}$ of genus 7, $\{5, 6\}$ and $\{6, 5\}$ of genus 3, $\{4, 7\}$ of genus 4, $r\left\{\begin{smallmatrix} 4 \\ 6 \end{smallmatrix}\right\}$ and $s\left\{\begin{smallmatrix} 6 \\ 6 \end{smallmatrix}\right\}$ of genus 2, $\left\{\begin{smallmatrix} 5 \\ 6 \end{smallmatrix}\right\}$ and $r\left\{\begin{smallmatrix} 5 \\ 6 \end{smallmatrix}\right\}$ of genus 3. *H. S. M. Coxeter* (Toronto, Ont.).

Deaux, R. Sur deux complexes quadratiques associés à un système de vecteurs. Mathesis 62, 102–110 (1953).

Scott, T. On the Φ-invariant of two quadrics. Proc. Edinburgh Math. Soc. (2) 10, 25–36 (1953).

The author discusses the necessity and sufficiency of the condition $\Phi \equiv (a_1'a_2'b_1'b_2')^2 = 0$ for the possibility of the construction of a tetrahedron self-polar to one of two quadrics $(a'x)^2=0$, $(b'x)^2=0$, and having its six edges tangent to the other. Evidently two systems of ∞^1 tetrahedra do exist. The author gives the equations of the locus of the vertices (being octavic curves) and the developable surfaces of class eight whose planes are the faces of the systems of tetrahedra.

E. M. Bruins (Baghdad).

Szász, Paul. Neue Herleitung der hyperbolischen Trigonometrie durch Verwendung der Grenzkugel. Acta Math. Acad. Sci. Hungar. 3 (1952), 327–333 (1953). (Russian summary)

By projecting a plane right-angled triangle onto a horosphere, the author obtains another variant of his method [Mat. Fiz. Lapok 48, 401–409 (1941); Acta Sci. Math. Szeged 12, Pars A, 44–52 (1950); these Rev. 8, 218; 12, 276] for deriving the classical formulas of hyperbolic trigonometry. *H. S. M. Coxeter* (Toronto, Ont.).

Rogačenko, V. F. On solvability of problems on construction in the Lobačevskiĭ plane by means of compass and hypocycle or orocycle and hypocycle. Doklady Akad. Nauk SSSR (N.S.) 88, 615–618 (1953). (Russian)

Smogorževskiĭ has shown [see Geometric construction in the Lobačevskiĭ plane, Gostehizdat, Moscow-Leningrad, 1951; these Rev. 14, 575] that any construction in the hyperbolic plane feasible by rule and compass, can also be accomplished by the compass and instruments that draw limit circles and equidistant curves. The present article shows that any two of these three instruments suffice.

H. Busemann (Los Angeles, Calif.).

Natucci, Alpinolo. Il teorema fondamentale della proiettività. Giorn. Mat. Battaglini (5) 1(81), 5–41 (1952).

An account is given of the different ways in which the fundamental theorem of projection geometry has been

demonstrated, in answer to a question raised in Boll. Un. Mat. Ital. 9, 114 (1930), question no. 39. This theorem can be based on three definitions of a projectivity between forms of the first order: a) they can be deduced from each other by a finite succession of projections and sections (Poncelet-Cremona); b) there exists a one-to-one correspondence which preserves the anharmonic ratio of four elements (Steiner-Chasles); c) there exists a one-to-one correspondence which preserves harmonic sets (von Staudt). Particular attention is paid to the work of Klein, Darboux, Reye, Bertini, Pieri, Enriques, Severi, Le Paige-Deruyts, Schur, Heffter, Cassina, and in particular of Comessatti. A final reference points to the work of Hodge and Pedoe; it also might have referred to that of Schreier and Sperner, and of Godeaux and Rozet. *D. J. Struik* (Cambridge, Mass.).

Lauffer, Rudolf. Zur Topologie der Konfiguration von Desargues. I. Math. Nachr. **9**, 235–240 (1953).

The ten lines of the general Desargues configuration decompose the real projective plane into 36 regions, consisting of 24 triangles and 12 quadrangles. F. W. Levi [Geometrische Konfigurationen, Hirzel, Leipzig, 1929, pp. 171 ff.] found six different types according to the way the regions are arranged. The types are most easily distinguished with reference to the twelve quadrangles, which may consist of (I) three blocks of four, (II) six pairs (each pair sharing a side), (III) four pairs and four single (each surrounded by four triangles), (IV) one block of four (not entirely surrounding their common vertex) and one pair and six single, (V) one block of four (surrounding their common vertex, as in case I) and eight single, or (VI) one block of six and six single. The author has invented a new way to describe these six types, making use of the signs of certain topological invariants. *H. S. M. Coxeter* (Toronto, Ont.).

Qvist, B. Some remarks concerning curves of the second degree in a finite plane. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **134**, 27 pp. (1952).

This paper is concerned with non-collinear sets of points in finite projective geometries, that is, sets with no three points on a line. Results of R. C. Bose [Sankhyā **8**, 107–166 (1947); these Rev. **10**, 201] and E. Seiden [Proc. Amer. Math. Soc. **1**, 282–286 (1950); these Rev. **11**, 679] are sharpened. In a finite plane with $n+1$ points on a line (not assumed Desarguesian) there are at most $n+2$ points in a non-collinear system and at most $n+1$ if n is odd. These systems are called curves, and indeed in a Desarguesian plane a non-degenerate conic section is an $n+1$ curve. A tangent to an $n+1$ curve is a line containing exactly one of its points. There is exactly one tangent through each point of an $n+1$ curve, while an $n+2$ curve has no tangents. If n is even, the tangents to an $n+1$ curve all pass through a single point which together with the curve form an $n+2$ curve. In three-space with $n+1$ points on a line a non-collinear system contains at most n^2+1 points, except in the single instance $n=2$, where the 8 points not on a plane form a non-collinear system. Quadric surfaces which are n^2+1 systems are constructed. *Marshall Hall, Jr.*

Berman, Gerald. Finite projective plane geometries and difference sets. Trans. Amer. Math. Soc. **74**, 492–499 (1953).

By a theorem of Singer's [same Trans. **43**, 377–385 (1938)] every finite Desarguesian plane is cyclic. If the plane is that coordinatized by $q=p^n$ elements, then a set of $q-1$ residues modulo N, with $N=q^2-q-1$, the residues d_i $(i=1, \cdots, q-1)$ have the property that d_i-d_j, $i\neq j$, yield all residues $\not\equiv 0$. Sets d_i and d_i-a are called equivalent. It is shown, following Singer's methods, that equivalent difference sets correspond to a particular primitive root in the field with q^3 elements. If $(t, N)=1$, then td_i will also be a difference set. Then td_i will be equivalent to d_i if and only if $t\equiv p^j$ (mod N). The discussion is confined to Desarguesian planes. *Marshall Hall, Jr.* (Columbus, Ohio).

***Järnefelt, G., and Kustaanheimo, Paul. An observation on finite geometries.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 166–182. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

In a geometry with coordinates from a field with a prime number of elements, p, the axioms of incidence will of course be satisfied. It is observed here that the quadratic form x^2-ky^2 with k a quadratic non-residue may be used to define a metric. Certain axioms of congruence are satisfied if this metric is used. It is conjectured that in a plane with p^2+p+1 points a set of $p+1$ points, no three on a line, will form a quadric. The reviewer finds this conjecture implausible. *Marshall Hall, Jr.* (Columbus, Ohio).

***Busemann, Herbert, and Kelly, Paul J. Projective geometry and projective metrics.** Academic Press Inc., New York, N. Y., 1953. viii+332 pp. $6.00.

The first two chapters contain a treatment of the real projective plane, using coordinates from the beginning and alternating between synthetic and analytic arguments. Because of the analogy with vectors, the authors use the multiplication sign to indicate both the join of two points and the meet of two lines (but on page 242 they tacitly admit the need for two distinct signs in the case of lines in space). A conic is defined as the class of self-conjugate points and lines in a hyperbolic polarity.

In Chapter III, the affine plane is derived by singling out (or omitting) one line, thus obtaining affine coordinates. There is an excellent treatment of convex sets, convex curves, and supporting lines. But it might well have been more clearly stated, on page 99, that affine geometry suffices for the definition of the area of a plane region in terms of an arbitrary unit area: there is no need to invent an "equiaffine" geometry. (Similarly, on page 278, it could have been said that affine geometry deals with ratios of lengths on parallel lines, ratios of areas in parallel planes, and ratios of volumes without restriction.)

Chapter IV is a highly original account of projective metrics, including perpendicularity and motions. Here a typical theorem is: If the circles are convex and the reflection in a line L exists and L' is perpendicular to L, then L is perpendicular to L'.

A "Minkowskian" metric is introduced into the affine plane by taking a closed, strictly convex, centrally symmetrical curve K, and calling it a unit circle. Such a Minkowskian geometry is said to be Euclidean if it admits the reflections in all the lines of one pencil (and therefore in all lines); this implies that K is an ellipse. The authors' definition of metric clearly excludes the "Minkowskian" geometry of the special theory of relativity, where the unit circle, being a hyperbola, is not convex [cf. Coxeter, Amer. Math. Monthly **50**, 217–228 (1943); these Rev. **4**, 226].

Another important metric is that of Hilbert, where the interior of a convex domain D is metrized by means of the logarithm of a cross ratio. Such a Hilbert geometry is said to be hyperbolic if it admits the reflections in all the lines of one pencil (and therefore in all secants of D); this implies

that D is the interior of an ellipse E. In Chapter V many of the classical properties of the hyperbolic plane are obtained by taking E to have the equation $x_1^2+x_2^2=1$, so that the line element is given by

$$ds^2=k^2\{dx_1^2+dx_2^2-(x_1dx_2-x_2dx_1)^2\}/(1-x_1^2-x_2^2)^2.$$

Later, the two non-homogeneous coordinates are replaced by three projective coordinates for which E is $x_1^2+x_2^2=x_3^2$, and these are normalized so that for all proper points $x_3^2-x_1^2-x_2^2=1$. Then analogous coordinates satisfying $x_1^2+x_2^2+x_3^2=1$ are used to define elliptic geometry.

In the final chapter most of the foregoing ideas are extended from two to three dimensions. It is proved, for instance, that a sphere of radius r in hyperbolic space is like a sphere of radius $k \sinh (r/k)$ in Euclidean space, from which it follows that the "limit sphere" (i.e., horosphere) has a Euclidean metric.

There are abundant exercises at the end of nearly every chapter. The book closes with a bibliography and an index. The printing has been well done, except that some pages are rather faint. Small misprints were noticed on pages 146 and 204.

H. S. M. Coxeter (Toronto, Ont.).

★Blumenthal, Leonard M. Theory and applications of distance geometry. Oxford, at the Clarendon Press, 1953. xi+347 pp. $10.00.

Since Menger's work [Math. Ann. **100**, 75–163 (1928)], the field known as metric topology or distance geometry (i.e., geometry of the subgroup of homeomorphisms for which the distance of two points is an invariant) has been greatly extended in content and application. A first survey was done by the author in his excellent volume on distance geometries [Univ. of Missouri Studies **13**, no. 2 (1938)]. The present book furnishes a detailed, self-contained and in many aspects a very complete treatment of the theory.

Part I deals with semimetric and metric spaces, starting from the preliminary notions and examples. The concepts of betweenness and convexity are introduced, as well as the definition and characteristic properties of metric segments. A chapter is devoted to curve theory in metric spaces; the concepts of arc length, geodesic arcs and different definitions of curvature and torsion are given and discussed. There is no reference to the work of Busemann on metric methods in Finsler spaces.

Part II deals with the distance geometry in Euclidean (E_n) and Hilbert $(\mathfrak{H})$ spaces. A fundamental problem of distance geometry is the so-called "space problem"; it seeks necessary and sufficient metric conditions that an arbitrary distance space of a specified class $\{S\}$ must satisfy in order that it may be congruent with a member of a given subclass of $\{S\}$. This problem is solved for the class of semimetric spaces and the subclass of the single space E_n and for the class of separable semimetric spaces and the subclass of the single $\mathfrak{H}$. The conditions are given in terms of the so-called Cayley-Menger determinants. Other noteworthy metric characterizations of E_n and $\mathfrak{H}$ are given; for instance, a finitely compact, convex, externally convex semimetric space with the weak euclidean four-point property is congruent with a euclidean space of finite dimension. Part II contains also the study of the congruence indices of some euclidean subsets (a regular polygon, a conic, a circle) with respect to the class of semimetric spaces and with respect to the class of subsets of E_n. A space R is said to have congruence indices (n, k) with respect to a class $\{S\}$ of spaces provided any space S of $\{S\}$ containing more than $n+k$ pairwise distinct points, is congruently imbeddable in R whenever each n of its points, not necessarily pairwise distinct, has that property. When the congruence indices are $(n, 0)$, n is said to be the "congruence order" of R with respect to $\{S\}$. A typical theorem proved in this direction is the following: the circular disc of radius r has congruence order three with respect to subsets of E_2.

Part III contains the distance geometry of the non-euclidean spaces (spherical $S_{n,r}$ of dimension n and radius r, elliptic $\mathcal{E}_{n,r}$ and hyperbolic $\mathcal{H}_{n,r}$). Some determinants, generalization of those of Cayley-Menger for euclidean spaces, play here a fundamental role in order to solve the space problem and some characterizations for such spaces. Some needed properties of these spaces, not readily found in the literature, are developed in the book. It is found that these properties do not characterize the non-euclidean spaces; they give rise to the definition of certain semimetric spaces with identical distance geometry to the non-euclidean spaces. For instance, the hyperbolic cosine function that is characteristic of hyperbolic space may be replaced by any one of a large class of functions without essential change in the distance geometry of the space. Some spherical theorems of Helly type are considered in detail, as well as the congruence indices of hemispheres and small caps. The metric geometry of the elliptic space $\mathcal{E}_{n,r}$ is particularly attractive and interesting. The necessary distinction between congruence and superposability gives rise to noteworthy questions referring to the existence of equilateral subsets in $\mathcal{E}_{n,r}$, solved only for $n\leq 3$. A chapter is devoted to the following problems: a) to find necessary and sufficient conditions for superposability of two congruent subsets of $\mathcal{E}_{n,r}$; b) to determine when a given congruence between two subsets of $\mathcal{E}_{n,r}$ may be extended to the whole space. The case of the elliptic plane is considered in detail; certain freely movable configurations are introduced which are useful in order to obtain the congruence order of $\mathcal{E}_{2,r}$ with respect to the class of all semimetric spaces. The analogous congruence order of $\mathcal{E}_{n,r}$ for $n>2$ is not known.

Part IV is devoted to the applications of distance geometry. Three illustrations are given: a) Applications to determinant theory (some properties of determinants of the Cayley-Menger type, difficult to prove in a purely algebraic way, are neatly proved when translated to its metrical meaning); b) Applications to sets of linear inequalities (provided by the results referring to congruence indices of hemispheres and small caps and by the intersection theorems for convex subsets of $S_{n,r}$); c) Applications to lattice theory (a normed lattice L may be made into a metric space $D(L)$ by attaching to each pair of elements x, y of L the "distance" $(x, y)=|x+y|-|xy|$; then, metric concepts and relations in $D(L)$ such as betweenness, congruent imbedding, . . . lead to concepts and relations in L). Applications to calculus of variations are not given.

The book is very well written; the exposition is lucid and the proofs are carefully arranged. It contains mainly the personal work of the author on the subject. Every chapter is well provided with exercises and concludes with references to the literature dealing with the material discussed. A course in general topology and a course in abstract algebra is all the preparation recommendable, though not indispensable, for a thorough understanding of the author's exposition.

L. A. Santaló (Buenos Aires).

Algebraic Geometry

Kivikoski, E. Zur Kennzeichnung der Kurven durch Singularitäten. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **131**, 21 pp. (1952).

A closed curve in the real projective plane which consists of a finite succession of convex arcs each point of which has a tangent is known as an elementary curve. This paper is a study of the effect on the structure of an elementary curve by various conditions imposed on its system of singularities. Known results are quoted. Among these is Kneser's theorem that no elementary curve exists all of the singularities of which are double points and also the earlier result of the author [Ann. Acad. Sci. Fennicae. Ser. A. **44**, no. 2 (1935)] that a curve with a single double point and no further singularities other than inflection points is of order 3. It is pointed out that these results are consequences of relations dealing with the numbers of singularities of an elementary curve given by J. v. Sz. Nagy [Math. Ann. **100**, 164–178 (1928)].

Curves are considered which possess at most one angle point of the first type and no further singularities other than cusps of the first type. These curves are shown to consist of 3 convex arcs. They are similar to curves of class 3 with 3 cusps and become such curves if no angle point is present. This result enables the author to determine the structure of elementary curves with one double point and no further singularities other than cusps of the first type. These curves are shown to be of two types one with 2 cusps, the other with 5 cusps; the order of curves of the first type is 4 or 6, that of curves of the second type 6 or 8.

Examples are given of curves whose only singularities are cusps of the second type. It is shown that such curves exist with only two cusps but of arbitrarily high order and also that such curves may have odd or even order and possess an arbitrarily high number of cusps. *D. Derry.*

Fusa, Carmelo. Alcune proprietà dei sistemi lineari di curve piane algebriche. Arch. Math. **3**, 465–469 (1952).

A theorem of M. Noether [Math. Ann. **3**, 161–226 (1870)] and one of O. Chisini [Atti Soc. Nat. Mat. Modena (5) **6**, 7–13 (1921)], both of which concern inequalities satisfied by the base multiplicities of a homaloidal net of plane algebraic curves are here extended to more general systems $|C_n|$ of degree d and genus p for which d and p satisfy certain inequalities. *H. T. Muhly* (Iowa City, Iowa).

Nagell, Trygve. Recherches sur l'arithmétique des cubiques planes du premier genre dans un domaine de rationalité quelconque. Nova Acta Soc. Sci. Upsaliensis (4) **15**, no. 6, 66 pp. (1952).

The present paper deals chiefly with elliptic plane cubics over an arbitrary commutative field, Ω. In the first of the two parts into which it is divided, a Weierstrass cubic $y^2=x^3-Ax-B$ is considered (A, B in Ω), together with the Abelian group G formed by its exceptional points, and any finite subgroup G_1 of G. Then the general expressions for A and B are obtained, when each of the exceptional points of G_1 is supposed to be rational (i.e., to have coordinates x, y in Ω) and G_1 has the order $n=3, 4, 5, 6, 7, 9, 15$; for $n=4, 9$, the two possibilities for G_1 to be cyclic or non-cyclic are investigated. The cases $n=3, 4, 5, 6, 7$ have already been studied by the author [Acta Math. **52**, 93–126 (1928); Nova Acta Soc. Sci. Upsaliensis (4) **14**, no. 1 (1946); Colloq. Internat. Centre Nat. Recherche Sci., no. **24**, 59–64 (1950); these Rev. **9**, 100; **12**, 852], and the proofs are here completed. The new case $n=15$ is particularly laborious and leads to a number of interesting incidental results, such as the following one: the plane quartic

$$u^3(v-2w)+u^2(-2v^2+4vw-w^2)\\ +u(v^3-3v^2w+4vw^2-2w^3)+vw(v-w)^2=0$$

has in $K(1)$ only six rational points, i.e., $(1, 1, 0)$, $(0, 1, 1)$, $(1, 1, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

In part II there are some properties of resolvent fields of a conic or cubic, and of rational triplets on a Weierstrass cubic. Then the equianharmonic cubics (1) $ax^3+by^3+cz^3=0$ and the cubics represented by the Hesse normal form (2) $x^3+y^3+z^3-3\alpha xyz=0$ are considered, and criteria for the linear and birational equivalence in Ω of two cubics of the same type (1) or (2) are given. For instance, in $K(1)$ there is an infinity of cubics $x^3+py^3+p^2z^3=0$, two-by-two non-equivalent; and, over a field not containing $\sqrt{-3}$, two cubics (2) are equivalent only if they have the same coefficient α. At the end, some calculations leading to a result stated in a previous paper [T. Nagell, Nova Acta Soc. Sci. Upsaliensis (4) **12**, no. 8 (1941); these Rev. **9**, 156] are sketched. *B. Segre* (Rome).

Marchionna, Ermanno. Caratterizzazione di curve gobbe segate da certe superficie secondo gruppi canonici. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15**(**84**), 139–142 (1951).

The author continues his study of the twisted curve in s_3 complete intersections of two surfaces of order μ, ν with a common multiple point [same Rend. (3) **14**(**83**), 137–158, 290–296 (1950); these Rev. **13**, 974, 975]. The theorem proved here is the following. Let Γ^* be a twisted curve of order $\mu\nu$ with an hk-ple point 0^*, Γ the plane curve of order $n=\mu\nu-hk$, projection of Γ^* from 0^*, and T the set of hk points of Γ arising in projection from 0. The necessary and sufficient condition that the canonical series on Γ^* may be traced by surfaces of order $m=\mu+\nu-4$ with a base point at 0^* of multiplicity $s=h+k-2$ is that the double points of Γ lie on a curve of order $l=(\mu-1)(\nu-1)-hk$ whose complete residual intersection with Γ is the set $(m-s)T$, and are thus $\delta=\frac{1}{2}[nl-hk(m-s)]$ in number. *P. Du Val.*

Amato, Vincenzo. Sulla costruzione delle equazioni delle curve G_8. Matematiche, Catania **7**, 62–66 (1952).

This note is an addendum to earlier notes [Matematiche, Catania **5**, 91–97 (1950); **6**, 113–118 (1951); these Rev. **12**, 389; **13**, 678]. A procedure for the determination of the equation of a curve with monodromy group G_8 is outlined. *H. T. Muhly* (Iowa City, Iowa).

Godeaux, L. Transformations birationnelles involutives laissant invariant le système des cubiques planes passant par six points fixes. Mathesis **62**, 85–89 (1953).

Godeaux, Lucien. Sur l'ordre d'une involution cyclique appartenant à une surface algébrique. Bull. Soc. Roy. Sci. Liège **22**, 77–84 (1953).

The author here more or less inverts the problem of an earlier paper [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° (2) **27**, no. 1626 (1952); these Rev. **14**, 578]. There the problem was: given an involution of prime order p, on a surface F, generated by a birational self-transformation of F, having an isolated united point A, simple in F, what is the nature of the corresponding point A' on a surface Φ, projective model of a linear system on F compounded with the involution and having no base point in A? It was shown

that the tangent cone to Φ at A' may have as many as four constituents, all rational and of orders a, m, n, b' (the notation is carried over from the memoir referred to); consecutive pairs have a common generator, and on each of these there may be a binode in the neighbourhood of A', with any number of partial neighbourhoods. In the present note it is shewn that if Φ has a singularity of this kind, and the three binodes have respectively u, t, v partial neighbourhoods each of grade -2, then the order of the involution is

$$p=[(t+1)mn+m+n][(u+1)a+1][(v+1)b'+1] +[(t+1)m+1][(u+1)a+1]b' +[(t+1)n+1][(v+1)b'+1]a+(t+1)ab'.$$

P. Du Val (Bristol).

Spampinato, Nicolò. Teoremi fondamentali sulle falde bidimensionali con l'origine in un punto o in una curva. Giorn. Mat. Battaglini (5) 1(81), 70–84 (1952).

Let a two-dimensional branch be represented in the complex projective space $S_3(x_1, \cdots, x_4)$ by the equations, $x_i=P_i(u, v)$, where P_i is a power series, and let P_{im} be the mth partial sum of P_i. The author discusses the rational surfaces, $x_i=P_{im}(u, v)$, when the origin of the branch is a point or a rational curve of order n, and the branch is otherwise "non-special". *H. T. Muhly.*

Todd, J. A. On the invariants of the canonical system of a V_4. Proc. Cambridge Philos. Soc. 49, 410–412 (1953).

The author has published (with E. A. Maxwell) a set of relations connecting the arithmetic genera of the i-dimensional characteristic systems of the canonical system on a V_4 [same Proc. 33, 438–443 (1937)]. These relations are formally different from a set of relations obtained earlier by Albanese [Ann. Mat. Pura Appl. (4) 4, 153–184 (1927)]. In this note the equivalence of the two sets of relations is established with the aid of the Newton-Stirling and Newton-Bessel interpolation formulas. *H. T. Muhly.*

Galafassi, Vittorio Emanuele. Omeomorfismi algebrici fra iperspazi reali. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 15(84), 130–138 (1951).

The topic of this paper is algebraic corresponding between two complex projective spaces S_r, S'_r, either birational or (α, β), which transform the real points of one into the real points of the other in a manner that is one-one without exception, and being continuous are thus topologically equivalent to collineations between the real spaces.

As regards Cremona transformations it is pointed out that every real primal of either homaloidal system must have one real sheet topologically equivalent to real projective S_{r-1}, and hence of odd order. The base manifolds must have no real points, and hence must be of even order. For $r=2$ the transformation of minimum order satisfying these conditions is that by quintics with six double base points, conjugate imaginary by pairs. For any odd value of $r=2p+1$ there is an involutory cubic transformation, in which the conditions for two points to correspond is that the line joining them meet two skew S_p's and that they be conjugate with respect to a fixed quadric; if the two S_p's are conjugate imaginary and the quadric has no real points, the transformation is of the kind envisaged. For even values of $r\geqq 4$ it is not known whether any such transformations exist.

For a correspondence which is (α, β) in the complex but one-one in the real field, since of the α (or β) points of one space that correspond to a given real point of the other just one must be real, α and β must both be odd. It is shown that for all r and for any odd values of α, β such transformations exist. The example given is the transformation in which corresponding points are projections from two points U, U^* in S_{r+1} of the same point of a primal Ψ of order α having an $(\alpha-\beta)$-ple point at U and not passing through U^*. If $\Psi\equiv\varphi F+tG$, where φ is a general real prime, F a real primal of order $\alpha-1$ with an $(\alpha-\beta)$-ple point at U and no other real point, an arbitrary real primal of order α with at least an $(\alpha-\beta)$-ple point at U, and t a sufficiently small real constant, every real line through U or U^* meets Ψ in only one real point, and the correspondence is obtained by a slight continuous deformation from the collineation in which Ψ is replaced by the prime φ.

Similar arguments shew that for any odd values of α, β it is possible to construct correspondences which are (α, β) in the complex field, but in the real field are topologically equivalent to any given Cremona transformation.

P. Du Val (Bristol).

Severi, Francesco. Le diverse concezioni di varietà nella geometria algebrica. Rend. Accad. Naz. dei XL (4) 2, 155–181 (1951).

A summary of this paper has been given by the author in two notes [C. R. Acad. Sci. Paris 232, 2395–2396; 233, 15–17 (1951); these Rev. 13, 65, 66]. The present, more detailed, treatment makes it possible to complete the cited reviews in several points. (a) The main object of the paper is to define the general concept of a base variety, independently of any previous analysis of the concept of infinitely near points. This object (which in classical geometry has never been achieved in all its generality) is satisfactorily achieved through the introduction of the concept of a complete polynomial ideal. (b) The author stipulates that his main definition of complete ideals (which is given in terms of intersection multiplicities with analytic branches) is not to be applied in the case of irrelevant H-ideals. For these latter ideals he chooses a different definition, namely, the following: every irrelevant H-ideal is complete. This definition has some undesirable consequences. For instance: the intersection of complete ideals is not necessarily complete; there are infinitely many "empty base varieties". (c) There are some deviations from common usage in the manner in which the author uses the terms "primary ideal", "irreducible ideal". These deviations lead the author to a correspondingly uncommon interpretation of the Lasker-E. Noether decomposition theorems for polynomial ideals, and as a consequence there is some doubt about the meaning or validity of Theorem VII. (d) Also Theorem VIII needs some clarification, for it has the following untenable consequence: every primary (non-irrelevant) ideal is complete. [The object described in (a) has been also one of the objects pursued by the reviewer [Amer. J. Math. 60, 151–204 (1938)]. In that paper, a polynomial ideal was defined to be "complete" if it is the intersection of valuation ideals. The author, in his definition, makes use only of zero-dimensional valuation (under the guise of analytic branches), while the reviewer has found it more convenient (for methodological reasons) to allow valuations of any dimension. The two definitions are easily seen to be equivalent. In an exchange of letters with the reviewer, the author has withdrawn his definition of complete irrelevant ideals as given in (b) above (accordingly, it is now the sense of the author's definition that there is only one complete irrelevant ideal, the unit ideal, and only one empty base variety) and has also stated that he will clarify in a subsequent paper the doubtful points referred to in (c) and (d).] *O. Zariski.*

van der Waerden, B. L. Zur algebraischen Geometrie. 16. Vielfältigkeiten von abstrakten Ketten. Math. Ann. 125 (1952), 314–324 (1953).

The author indicates how it is possible to define the notion of a specialization of a (positive) cycle C on an abstract variety V (in the sense of A. Weil). Let V be given by means of a collection of varieties V_a in projective spaces, on each one of which there is given a frontier F_a; let T_{ba} be the birational correspondence between V_a and V_b which is used in the definition of V. We may assume without loss of generality that C is a subvariety of V; let C_a be the subvarieties of the varieties V_a which define C (on those V_a such that C is represented on V_a). Assume that the varieties C_a, considered as cycles in projective spaces, are simultaneously specialized to cycles D_a'; for each a, let D_a be the cycle obtained from D_a' by omitting those components which are on F_a. Then the main lemma says that, for any pair (a, b) such that C_a, C_b exist and for any variety A_a in V_a which occurs in D_a, the variety A_b on V_b which corresponds to A_a by T_{ba} occurs the same number of times in D_b as A_a in D_a. In order to prove this, the author considers the varieties C_{ba} which represent (in suitable projective spaces) the restrictions of the correspondences T_{ba} to the varieties C_a, and he extends the specialization

$$(\cdots, C_a, \cdots) \rightarrow (\cdots, D_a', \cdots)$$

to a specialization $(\cdots, C_{ba}, \cdots) \rightarrow (\cdots, D_{ba}, \cdots)$ of these varieties. Then he proves that, if B_{ba} is the representative variety of the restriction of T_{ba} to A_a, then A_a occurs as many times in D_a as B_{ba} in D_{ba}. This in turn is accomplished by a reduction to the 0-dimensional case by intersecting with generic projective varieties. Some of the steps in the argument are rather perfunctorily treated; it seems to the reviewer that, in view of the importance of the question, a more formal proof would be in order. *C. Chevalley.*

Rosenlicht, Maxwell. Differentials of the second kind for algebraic function fields of one variable. Ann. of Math. (2) 57, 517–523 (1953).

Let R be a field of algebraic functions of one variable over a field K. There was up to now a certain doubt as to how differentials of the second kind should be defined in R when K is of characteristic $\neq 0$. The author shows conclusively that the correct definition is the following: ω is of the second kind if, for every place $\mathfrak{p}$, there exists an $f_{\mathfrak{p}} \in R$ such that $\omega - df_{\mathfrak{p}}$ has no pole at $\mathfrak{p}$. Let d be the dimension of the space of differentials of the second kind modulo exact differentials; it is well known that $d = 2g$ (where g is the genus of R) if K is of characteristic 0. If not, then the following results are established: if R is separably generated and if its genus cannot be lowered by an extension of the basic field, then $d = g$; if R is separably generated and if its genus can be lowered to g' under an algebraically closed extension of the basic field (with $g' < g$), then $g' \leqq d < g$; if R is not separably generated, then $d = g$. Let S be a field of algebraic functions of one variable which contains R and has the same field of constants K as R. Then, if ω is of the second kind in R, $\mathrm{Cosp}_{R/S}\,\omega$ is of the second kind in S, while, if Ω is of the second kind in S, then $\mathrm{Sp}_{S/R}\,\Omega$ is of the second kind in R.

The main justification of the definition lies in certain results which are announced without proof in this paper. Assume that K is algebraically closed and that C is a model of R without singularity. Then it is possible to imbed C rationally into certain "generalized Jacobian varieties" $J_{\mathfrak{o}}$ which depend on semi-local subrings $\mathfrak{o}$ of R. $J_{\mathfrak{o}}$ is a group variety and contains a linear algebraic subgroup H such that $J_{\mathfrak{o}}/H$ is the Jacobian variety of R. In order for H to be the product of a certain number of times the additive group K_+ of K, it is necessary and sufficient that the differentials induced on C by the group invariant differentials of $J_{\mathfrak{o}}$ be of the second kind. If we require that $J_{\mathfrak{o}}$ should not split into the direct product of a group and a certain number of groups of type K_+, then $J_{\mathfrak{o}}$ must be of dimension $\leqq 2g$, and this maximum is attained (for any characteristic). The fact that the space of differentials of the second kind modulo exact differentials is of dimension g only in characteristic p is attributed to the fact that K_+ then admits proper rational homomorphisms. *C. Chevalley.*

★Samuel, Pierre. Algèbre locale. Mémor. Sci. Math., no. 123. Gauthier-Villars, Paris, 1953. 76 pp. 950 francs.

This is a survey of the technical algebraic devices which are used in the local study of an algebraic variety in the neighbourhood of one of its points or of a subvariety. These devices have been developed by various authors: W. Krull, who introduced the fundamental notion of a local ring, O. Zariski, I. Cohen, P. Samuel, and C. Chevalley. Their results are scattered among various papers, a circumstance which made it difficult up to now to obtain a general view of the subject. This monograph contains a systematic and logical presentation of these results, together with some new improvements.

Chapter I is concerned with $\mathfrak{m}$-adic rings and Zariski rings. An $\mathfrak{m}$-adic ring is a Noetherian topological ring A in which a fundamental system of neighbourhoods of 0 is formed by the powers of an ideal $\mathfrak{m}$, with the assumption that this topology should be a Hausdorff topology (i.e., that no element of $1+\mathfrak{m}$ should be a zero divisor). If it is further assumed that every ideal in A is closed, then A is called a Zariski ring. A semi-local ring (i.e., a Noetherian ring with only a finite number of maximal ideals) is a Zariski ring; this applies in particular to local rings (i.e., Noetherian rings with only one maximal ideal). It should be observed that the author requires in the definition of a Zariski ring A that A should not be discrete; this requirement seems unfortunate, since a field should be considered as a special case of a local ring and therefore of a Zariski ring. The main operations which have to be performed on $\mathfrak{m}$-adic rings are the following: completion, construction of factor rings, construction of rings of quotients, finite extensions. The effect of these operations is analyzed carefully by the author.

Chapter II is concerned with characteristic polynomials. The main result is the following far-reaching generalization (due to the author) of a well-known theorem of Hilbert: if A is a semi-local ring and $\mathfrak{v}$ a defining ideal of A (i.e., an ideal whose powers define the topology of A), then the length of the ideal $\mathfrak{v}^n$ is, for n large enough, a polynomial $P(n)$ in n. The degree of this polynomial, which does not depend on $\mathfrak{v}$, is called the dimension of A; this is also the smallest integer d such that some set of d elements generates a defining ideal of A; for a local ring, d is the largest length of a chain of prime ideals in A. If $e(d!)^{-1}n^d$ is the leading term of $P(n)$, then e is an integer, which is called the multiplicity of $\mathfrak{v}$. This definition is equivalent to the one given by the reviewer for the case of a local ring, and which is the base of the notion of intersection multiplicity for algebroid varieties.

Chapter III is concerned with geometric local rings, i.e., with those local rings which occur in algebraic geometry. These rings have some very useful special properties (analytical equidimensionality, analytical non-ramification,

transition theorem, associativity formula), which do not belong to the most general local rings. Their definition (due to the reviewer) is unfortunately most artificial and unnatural; it is to be hoped that this class of rings can be characterized in a more intrinsic manner in the future.

Chapter IV is concerned with the structure of complete local rings. The theory of these rings (essentially due to I. Cohen) is made to depend primarily on a "lifting theorem", due to I. Cohen, but which played a very subordinate role in this author's exposition. The structure of complete local rings is completely known only in the case where the ring A under consideration is regular (i.e., the maximal prime ideal has a system of d generators, if d is the dimension) and, in the case of unequal characteristics, is unramified (if $\mathfrak{m}$ is the maximal ideal, then the case of unequal characteristics is the one in which A is of characteristic 0 but $A/\mathfrak{m}$ of characteristic $p>0$; A is then called unramified if p does not belong to $\mathfrak{m}^2$). In these cases, A is a ring of formal power series either over a field or over a complete valuation ring whose valuation ideal is generated by p.

Chapter V is concerned with local rings which are either integrally closed or factorial (a factorial ring is a ring in which the unique factorization theorem of elements into primes is true). The main result (due to O. Zariski) states that, if a geometric local ring has no zero divisor and is integrally closed, then its completion still has the same properties.

Chapter VI is concerned with miscellaneous complements. This little book contains, in a condensed form, an encyclopedic repertorium of almost everything which is known on the subject. Sufficient indications of the proofs are given to make it easy to reconstruct them entirely. The nature of the applications of the various theorems to algebraic geometry is in general rapidly explained. There is unfortunately no index, with the effect that it is sometimes a little difficult to locate the exact definition of a term. *C. Chevalley.*

Differential Geometry

Goormaghtigh, R. Sur un problème de géométrie infinitésimale. Mathesis **62**, 99–102 (1953).

Jha, P. On curves having a given curve for the locus of the centre of spherical curvature. Math. Student **20**, (1952), 115–118 (1953).

Marussi, Antonio. Le coordinate intrinseche della geodesia. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 111–120 (1952).

In order to deal more effectively with broad areas of geodesy, and in particular with extreme altitudes in a spatial geodesy, it is proposed to refer to the set of potential levels defined by the earth's gravitational field. This gravitational potential represents a third geodesic coordinate along with conventional latitude and longitude. The foundations and general functional relations for the differential geometry of such a coordinate system are presented in detail.

N. A. Hall (Minneapolis, Minn.).

Gifford, P. W., Jr. Some refinements in the theory of specialized space curves. Amer. Math. Monthly **60**, 384–393 (1953).

The usual necessary and sufficient conditions that a curve be a straight line (or a plane curve) assume that $\mathbf{x}'(t)\neq 0$ (or that $\mathbf{x}'(t)\times\mathbf{x}''(t)\neq 0$). There are many parameterized straight lines (or plane curves) which do not satisfy these requirements. This paper replaces these with weaker hypotheses which still permit the conclusion that certain curves are straight lines (or plane curves). Typical theorems are: If $\mathbf{x}(t)$ is analytic, $\mathbf{x}'(t)\neq 0$, $\mathbf{x}'(t)\times\mathbf{x}''(t)\equiv 0$, then the curve is a straight line. If $\mathbf{x}(t)$ is analytic, $\mathbf{x}'(t)\times\mathbf{x}''(t)\neq 0$, $(\mathbf{x}'\mathbf{x}''\mathbf{x}''')\equiv 0$, the curve is plane. The non-analytic case is also treated. *C. B. Allendoerfer* (Seattle, Wash.).

Gołąb, S. Sur une condition nécessaire et suffisante d'ombilicité d'un point de surface. Ann. Soc. Polon. Math. **25** (1952), 140–144 (1953).

The author proves that a necessary and sufficient condition that a point on a surface be an umbilical point is that the normal curvatures of the two lines of curvature through this point be equal. *C. B. Allendoerfer* (Seattle, Wash.).

Kruppa, Erwin. Zum Dualitätsprinzip in der Differentialgeometrie dritter Ordnung. Arch. Math. **3**, 401–408 (1952).

Let P be a point on an analytic surface Φ. The principal curvatures $1/R_1$ and $1/R_2$ of Φ at P are assumed not to vanish. In a suitable coordinate system with the origin P, Φ will have a Taylor expansion $z=\phi_2(x,y)+\phi_3(x,y)+\cdots$. Here $\phi_2=\frac{1}{2}(x^2/R_1+y^2/R_2)$ and $\phi_3(x,y)$ is a homogeneous cubic polynomial which is assumed not to vanish identically. The inhomogeneous coordinates u, v, w of the tangent planes $z=ux+vy+w$ of Φ near P have the Taylor expansion $w=-\phi_2(R_1u,R_2v)+\phi_3(R_1u,R_2v)+\cdots$. The cubic indicatrix $i^3: \phi_2(x,y)+\phi_3(x,y)=0$ was studied by Groiss and Kruppa [Akad. Wiss. Wien, S.-B. IIa, **156**, 9–48 (1948); these Rev. **9**, 464]. In the present paper, an indicatrix cone κ_3^4 and a dual cubic indicatrix i_d^3 are introduced. κ_3^4 is the rational cone $-\phi_2(R_1u,R_2v)+\phi_3(R_1u,R_2v)=0$ of order four and class three; i_d^3 is the rational cubic $\phi_2(R_1x,R_2y)+\phi_3(R_1x,R_2y)=0$. The following topics, among others, are discussed: Affine geometric interpretations of κ_3^4 and i_d^3; constructions of the affine normal of Φ at P by means of i_d^3 and i^3; regular quadrics; Darboux and Segre tangents. *P. Scherk.*

Wunderlich, Walter. Über die Torusloxodromen. Monatsh. Math. **56**, 313–334 (1952).

Let $z^2+((x^2+y^2)^{1/2}-a)^2=b^2$ be the equation of the torus Φ in rectangular cartesian coordinates. If $a<b$, then an inversion with respect to a sphere with the center $(0,0,(b^2-a^2)^{1/2})$ transforms Φ into a right circular cone and the loxodromes Λ of Φ into the well-known loxodromes of the cone. This leads to a parametric representation of the Λ's which implies, e.g., that the real Λ's are transcendental curves with the asymptotic points $(0,0,\pm(b^2-a^2)^{1/2})$.

The case $a>b$ is obtained from the preceding one by a complex transformation which yields the parametric representation

$$(1)\quad \mathfrak{x}=\frac{c}{a-b\cos p\phi}(c\cos\phi,\, c\sin\phi,\, b\sin p\phi),\quad c=(a^2-b^2)^{1/2}$$

of the Λ's through the point $P\equiv(a+b,0,0)$. The parameter p distinguishes between the various Λ's; $p=\pm 1$ yields the two Villarceau circles through P. Λ is algebraic$\leftrightarrow\Lambda$ is closed$\leftrightarrow p$ is rational. (1) is connected with the parametric representation $\mathfrak{x}\equiv c(a-b\cos\psi)^{-1}(c\cos\phi,\, c\sin\phi,\, b\sin\psi)$ of Φ and with a cinematic generation of the Λ's by means of Bennett's isogram [Proc. London Math. Soc. (2) **13**, 151–173 (1914)]. Φ is mapped through $\xi=c\phi$, $\eta=b\psi$ conformally

into the (ξ, η)-plane, the images of the Λ's being the straight lines. Among the numerous other results one more may be quoted: The Λ's are identical with the Darboux curves of Φ, i.e., with those curves whose osculating spheres are tangent spheres of Φ. *P. Scherk* (Saskatoon, Sask.).

Wunderlich, Walter. Beitrag zur Kenntnis der Minimalschraubflächen. Compositio Math. **10**, 297–311 (1952).

The author presents a simple proof that the isotropic curves of a minimal helicoid are right circular helices. This yields a fast approach to a well-known parametric representation of these helicoids [Graustein, Differential geometry, Macmillan, New York, 1935, p. 226] which is the basis of a detailed discussion. The following sample may be sufficient: If the cylinder $x=a \cosh z/a$ is screwed about the z-axis, then it envelops a minimal helicoid.
P. Scherk (Saskatoon, Sask.).

Lauffer, Rudolf. Wege in Minimalebenen. Math. Nachr. **9**, 241–242 (1953).

The author develops formulas for the lengths of curves on minimal planes. Applications are made to a set of $2n$ minimal planes. Relationships are developed among the lengths of certain important segments, and among corresponding vectors. *C. B. Allendoerfer* (Seattle, Wash.).

Ancochea, Germán. Sur les formes différentielles quadratiques dégénérées. C. R. Acad. Sci. Paris **236**, 2205–2207 (1953).

Consider the form $F=g_{ij}dx^idx^j$ $(i, j=1, \cdots, n)$. Let

$$\gamma_{irs}=\frac{1}{2}\left(\frac{\partial g_{is}}{\partial x^r}-\frac{\partial g_{ir}}{\partial x^s}\right).$$

It is proved that the minimum number of variables by means of which this form can be expressed is the rank of the matrix $\left\|\begin{matrix} g_{ij} \\ \gamma_{rsj} \end{matrix}\right\|$. F can also be written: $F=(\omega^1)^2+\cdots+(\omega^h)^2$ $(h\leqq n)$. Let $\bar{\omega}^a$ be $n-h$ other forms such that the system: $\omega^i, \bar{\omega}^a$ has rank n. Write

$$d\omega^i=\gamma^i{}_{rs}[\omega^r\omega^s]+\gamma^i{}_{j\alpha}[\omega^j\bar{\omega}^\alpha]+\gamma^i{}_{\beta\alpha}[\bar{\omega}^\beta\bar{\omega}^\alpha]$$

$(i, j, r, s=1, \cdots, h; \alpha, \beta=1, \cdots, n-h)$. Then the minimum number of variables above is also equal to the rank of the matrix: $\left\|\begin{matrix} \gamma^i{}_{j\alpha} \\ \gamma^i{}_{\beta\alpha} \end{matrix}\right\|$. *C. B. Allendoerfer* (Seattle, Wash.).

Dubnov, Ya. S. Diagonal properties of nets. Trudy Sem. Vektor. Tenzor. Analizu **9**, 7–48 (1952). (Russian)

An equation $\varphi_{\alpha\beta}du^\alpha du^\beta=0$, $\alpha, \beta, i=1, 2$ on a surface determines two congruences with tangent vectors Λ^i, M^i and equations $u(u^1, u^2)=\text{const.}$, $v(u^1, u^2)=\text{const.}$ A third congruence N^i forms with the other two a diagonal set $w(u^1, u^2)=\text{const.}$, if these functions exist, such that

$$f(u)+g(v)+h(w)=\text{const.}$$

A necessary and sufficient condition is that the scalar field

$$\frac{E^{\lambda\mu}(\phi_\lambda{}^\alpha w_\alpha)_{|\mu}}{E^{\lambda\mu}\phi_\lambda{}^\alpha w_\alpha w_\mu}$$

has the curves $w=\text{const.}$ as level lines. Here $E_{\alpha\beta}=-E_{\beta\alpha}$ and $\phi_\lambda{}^\alpha$ is the "unit deviator"

$$\phi_j{}^i=\frac{E^{i\alpha}\varphi_{\alpha j}}{(-\frac{1}{2}E^{\alpha\alpha_1}E^{\beta\beta_1}\varphi_{\alpha\beta}\varphi_{\alpha_1\beta_1})^{1/2}} \quad \text{(all indices from 1 to 2).}$$

A scalar field w is called additive-diagonal with respect to the set given by $u=\text{const.}$ and $v=\text{const.}$ if for a curvilinear quadrangle $ABCD$ (A, C opposite vertices)

$$w(A)+w(C)=w(B)+w(D),$$

and multiplicative-diagonal if $w(A)w(C)=w(B)w(D)$. Conditions for these cases are derived, and, after a Riemannian metric has been introduced, a number of special cases (rhombic sets, isothermic sets, potential sets, etc.) are investigated. *D. J. Struik* (Cambridge, Mass.).

Hartman, Philip, and Wintner, Aurel. On the singularities in nets of curves defined by differential equations. Amer. J. Math. **75**, 277–297 (1953).

The behavior of solution paths of $a\,dx^2+2b\,dxdy+c\,dy^2=0$ near $(0, 0)$ is investigated, assuming that the coefficient functions a, b, c are continuous near $(0, 0)$ differentiable at $(0, 0)$, with $a=b=c=0$ at $(0, 0)$ and (1) $b^2-ac>0$ for $x^2+y^2>0$. (1) implies that solution paths near $(0, 0)$ can be divided into two well-defined sets, and the main theorem states sufficient conditions on a, b, c in order that each set shall contain at least one solution path tending to $(0, 0)$, and that any such path shall satisfy $\tan^{-1}(y/x)\to\theta_0$, $\tan^{-1}(dy/dx)\to\theta_0 \pmod{\pi}$, for some θ_0. These conditions, too lengthy to state here, involve only the partial derivatives of a, b, c at $(0, 0)$, and are certainly satisfied if (1) is strengthened to $b^2-ac\geqq k(x^2+y^2)$, $k>0$. The main theorem is used to describe the behavior (i) of lines of curvature on a surface near an isolated umbilical point and (ii) of asymptotic lines near an isolated flat point on a surface of non-positive curvature. *G. E. H. Reuter* (Manchester).

Bompiani, Enrico. Sulle coordinate di Grassmann. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 329–335 (1952).

The Grassmann coordinates of a linear space P_k in a projective space P_n satisfy a number of relations, the most simple of which is the quadratic relation

$$(1) \qquad p^{j_1\cdots j_k[i}p^{hl]j_1\cdots j_{k-1}}=0.$$

The author shows that this relation is sufficient to compute all the coordinates if the $p^{j_1\cdots j_k i}$ $(j_1, \cdots, j_k\subset 0, 1, \cdots, k)$ are given and $p^{01\cdots k}\neq 0$. It is shown that the coordinates which cannot be computed directly from (1)—because the corresponding equation is an identity—are zero. Furthermore, necessary and sufficient conditions are given in order that a P_k and a P_h intersect in a P_t. The Grassmann coordinates of P_t are given in terms of those of P_k and P_h.
J. Haantjes (Leiden).

Müller, Hans Robert. Flächenläufige Bewegungsvorgänge im elliptischen Raum. I. Monatsh. Math. **57**, 29–43 (1953).

Eingliedrige Bewegungen im elliptischen Raum sind von Blaschke und Garnier untersucht worden; der Verfasser betrachtet Bewegungsvorgänge welche von zwei Parametern abhängen. Über die Methode ist folgendes zu sagen: 1) für die analytische Bestimmung der Bewegung werden Quaternionen angewandt; 2) durch die sphärische Abbildung von Study werden den orientierten Geraden des Raumes ein Paar Einheitsvektoren, d.h. die Punkte von zwei Bildkugeln zugeordnet; 3) neben dem "Gangtetraeder" und dem "Rasttetraeder" wird (nach dem Beispiel Van der Woude's) ein "Bezugstetraeder" benutzt, über dessen nähere Festlegung man noch jeweils verfügen kann. Während zu jeder Stelle einer eingliedrigen Bewegung ein Paar Momentanachsen und ein Komplex gehört, findet Verfasser bei der flächenläufigen Bewegung zwei "Polach-

sen", sämtliche Momentanachsen schneiden sie rechtwinklig, man hat jetzt momentan ein Komplexbüschel. Von Bedeutung ist der Ort der Polachsen in Gang- und Rastraum; sie sind so aufeinander bezogen dass für jedes Paar entsprechender geschlossener Regelflächen Übereinstimmung gewisser Integralinvarianten ("Öffnungsstrecke" und "Öffnungswinkel") besteht. Umgekehrt können zwei "öffnungstreue" Strahlensysteme immer als die Örter von Polachsen eines Bewegungsvorganges betrachtet werden. Untersucht wird noch die Gesamtheit aller Geraden des Gangraumes, die bezüglich sämtlicher Schrauben des Büschels der Momentanschrauben ein verschwindendes Moment besitzen.

O. Bottema (Delft).

Petkantschin, B. Isometrie zwischen zwei Regelflächen mit isotropen Richtebenen. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie I. **47**, 139–155 (1951). (Bulgarian. German summary)

Petkantschin, B. Über die Zentralkurve einer Regelschar mit isotroper Richtebene. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie I. **47**, 107–138 (1951). (Bulgarian. German summary)

Pedersen, Flemming P. On spaces with negative curvature. Mat. Tidsskr. B. **1952**, 66–89 (1952).

H. Busemann has introduced and studied certain metric spaces (G-spaces) with non-positive curvature [Acta Math. **80**, 259–310 (1948); these Rev. **10**, 623]. The purpose of the present paper is to show that many of Busemann's results hold with the following more general definition of non-positive curvature: a G-space R is said to have non-positive (negative) curvature if every point p of R has a spherical neighborhood $S(p, r_p)$ such that: a) for any pair $x, y \in S$ the segment $T(x, y)$ is unique; b) for any two (non-collinear) segments T_1, T_2 in S, the distances $x(t)T_2$ and $y(t)T_1$ are (strictly) semi-convex functions, where $x(t) \in T_1$, $y(t) \in T_2$. If T denotes a segment and $C_T{}^\alpha$ is the locus of points x such that $xT \leqq \alpha$, the preceding definition has the following geometric interpretation: a G-space R has non-positive (negative) curvature if and only if $C_T{}^\alpha \subset S(p, \eta(p))$ is convex (strictly convex) for every point p of R; $\eta(p)$ is the least upper bound of those r for which any segment T in $S(p, r)$ is subsegment of a segment T' with the same center as T but twice as long. For straight spaces (i.e., spaces in which all geodesics L are congruent to a euclidean straight line) of non-positive curvature, the sets $C_L{}^\alpha$ are convex for all L and α. Following the work of Busemann, some properties of the universal covering space of R, the theory of parallel lines, motions without fixed points and the behavior of geodesics are considered with the new definition of non-positive curvature. Finally it is shown that in the case of Riemannian spaces that definition coincides with the usual one. For Finsler spaces it is shown that if a two-dimensional Finsler space has non-positive curvature in the present sense, then the Finsler curvature as usually defined is non-positive. It remains an open question whether or not the converse holds.

L. A. Santaló (Buenos Aires).

Rauch, H. E. On differential geometry in the large. Proc. Nat. Acad. Sci. U. S. A. **39**, 440–442 (1953).

The following result is announced: Given any compact simply connected symmetric space $\mathcal{S}^n$ with holonomy group $\mathfrak{K}$, then any Riemannian M^n, with holonomy group $H \subseteq \mathfrak{K}$, whose curvature differs from that of $\mathcal{S}^n$ at most by numerical limits depending only on $\mathcal{S}^n$ has universal covering $\tilde{M}^n$ homeomorphic with $\mathcal{S}^n$.

C. B. Allendoerfer.

Pogorelov, A. V. On extrinsic curvature of smooth surfaces. Doklady Akad. Nauk SSSR (N.S.) **89**, 407–409 (1953). (Russian)

Let $\mathfrak{r}(u, v)$ represent vectorially a surface S of class C' in E^3 with $\mathfrak{r}_u \times \mathfrak{r}_v \neq 0$. Then every closed set F on S has a closed spherical image whose measure we denote by $\sigma(F)$. For any open set G on S the extrinsic absolute curvature is defined by $\sigma^0(G) = \sup_{(F_j)} \sum_{j=1}^n \sigma(F_j)$, where $(F_1, \cdots, F_n) = (F_j)$ traverses all finite sets of disjoint closed sets in G. For an arbitrary set M on S we put $\sigma^0(M) = \inf_{M \subset G} \sigma^0(G)$. The surface S is divided into two sets ϕ^+ and ϕ^-. The first, ϕ^+, consists of those points p where a suitable neighborhood of p on S lies entirely in one of the closed halfspaces bounded by the tangent plane of S at p, and $\phi^- = S - \phi^+$. Put $\sigma^+(M) = \sigma^0(M \cap \phi^+)$, $\sigma^-(M) = \sigma^0(M \cap \phi^-)$, $\sigma(M) = \sigma^+(M) - \sigma^-(M)$. The surface S is said to have bounded extrinsic curvature if every point of S has a neighborhood G for which $\sigma^0(G)$ is finite. Then σ^0, σ^+, σ^-, σ are finite for every compact subset of S and are completely additive set functions on the Borel sets of S. The main result, stated without proof, is this: If S has bounded extrinsic curvature, then a suitable neighborhood of a given point of S can be uniformly approximated by analytic surfaces whose Gaussian curvatures are uniformly bounded. It follows that the surfaces of bounded extrinsic curvature coincide with the surfaces of bounded (intrinsic) curvature of A. D. Alexandrov.

H. Busemann.

Graiff, Franca. Sull'integrazione tensoriale negli spazi di Riemann a curvatura costante. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 155–163 (1951).

The problem considered is the solution of equations of the form

$$(*) \qquad U_{h_1 \cdots h_r/l} = T_{h_1 \cdots h_r l},$$

where U is unknown, and where T is given. The first conditions of integrability are a set of relations between $U_{h_1 \cdots h_r}$ and $T_{h_1 \cdots h_r j/m}$, and because the curvature tensor has a very simple form, the problem can be reduced to a similar one for r replaced by $r-2$. For $r=0$ the solution is obtained by integration; while for $r=1$ the U_h can be solved from the integrability conditions, and substitution in (*) thus gives a true or false relation between T and its second covariant derivatives, which determines whether or not the system (*) admits a solution.

A. Nijenhuis (Princeton, N. J.).

Hartman, Philip, and Wintner, Aurel. On the existence of Riemannian manifolds which cannot carry non-constant analytic or harmonic functions in the small. Amer. J. Math. **75**, 260–276 (1953).

Les auteurs désignent par C^n la classe des fonctions n fois continûment différentiables sur un ouvert D et par $C^n(\lambda)$ la classe de ces fonctions pour lesquelles les dérivées d'ordre n satisfont à une condition de Hölder uniforme d'exposant λ $(0 \leqq \lambda \leqq 1)$ sur tout compact de D. Pour $n=0$ on a les fonctions supposées seulement continues. Ils étudient les C^n [ou $C^n(\lambda)$]-métriques riemanniennes définies sur un disque ouvert à 2 dimensions D et en particulier la classe C_0 des métriques "exactement continues" qui sont C^0 mais ne sont isométriques avec aucune C^1-métrique par une C^1-transformation, ou la classe C_n des C^n-métriques qui ne sont isométriques avec aucune C^{n+1}-métrique. L'existence de métriques appartenant aux classes C_0 ou C_n est ici prouvée.

Les principaux résultats démontrés sont les suivants qui sont en relation avec un théorème de Lichtenstein [Bull. Internat. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. Sér. A. Sci. Math. **1916**, 192–217]: on peut trouver des C^0-mé-

triques sur le disque D, telles qu'il n'existe, aussi petit que soit D, aucune C^1-transformation à Jacobien non nul qui amène la métrique à la forme conforme à un ds^2 euclidien. [Le résultat de Lichtenstein entraîne la conclusion inverse pour une $C^0(\lambda)$-métrique $(\lambda\neq 0)$.] Il existe des C^0-métriques pour lesquelles le système des équations de Cauchy-Riemann associées,

$$gv_x = g_{12}u_x - g_{11}u_y, \quad gv_y = g_{22}u_x - g_{12}u_y,$$

n'admet pour C^1-solutions que des constantes. Pour une telle métrique, le problème de Dirichlet, relatif à un ouvert B limité par une courbe de Jordan $\dot{B}$ et à une donnée frontière continue non constante, n'admet pas dans B de C^1-solution continue dans $B+\dot{B}$.

Les résultats sont étendus à des C^n-métriques sous la forme suivante: il existe des C^n-métriques telles qu'il n'existe, aussi petit que soit D, aucune C^{n+1}-transformation à Jacobien non nul (il en existe de classe C^n d'après Lichtenstein) qui amène la métrique à la forme conforme à un ds^2 euclidien. En particulier, il existe des C^1-métriques pour lesquelles il n'existe aucune fonction harmonique non constante (en entendant par fonction harmonique une C^2-solution de l'équation de Laplace relative à la métrique).

A. Lichnerowicz (Paris).

Craig, Homer V., and Townsend, B. B. On certain metric extensors. Pacific J. Math. 3, 25–46 (1953).

The author shows how we can find the metric extensors of the higher space derived from a Riemannian space R_N by making use of the representation of R_N on a flat vector space by means of the radius-vector $\mathfrak{p}(x^1, x^2, \cdots, x^N)$. In fact, dot product operation (i.e., scalar product) and successive differentiation in the immersed space produce the metric extensors which contain among their components the fundamental metric tensor and associated two-index components of connection $[ab, c]x^{b'}$ and $\{{a \atop bc}\}x^{b'}$: the direct connection extensor $g^b{}_{aa}$, the alternation connection extensor $L^b{}_{aa}$ and $g_{aa\beta b}$, and their interchanges $g_b{}^{aa}$, $L_b{}^{aa}$, $g^{aa\beta b}$. The latter ones can be related to the former ones by means of the interchange extensors $\mathfrak{g}_{aa\beta b}$, $\mathfrak{g}^{aa\beta b}$ expressed by the higher order derivatives of g_{ab}. Next it is shown that we can make extensors from the derivatives of equipollent tensors by means of $L^b{}_{aa}$. Finally, the tensor $D_{bcd}^{\cdots a} = \mathfrak{p}_{bc}\cdot\mathfrak{p}_d{}^a + \{{e \atop bc}\}\{{a \atop ed}\}$ is obtained, computing the alternating intrinsic derivative of a vector, where $\mathfrak{p}_{bc} = \partial^2\mathfrak{p}/\partial x^b\partial x^c$ and $\mathfrak{p}_d{}^a = \partial(g^{ab}\mathfrak{p}_b)/\partial x^d$. It should be noticed that in the theory one can refer not only to curves, i.e., a single parameter, but also to two-dimensional spreads involving two parameters, as the authors use Craig's notation of "matrix primes".

A. Kawaguchi.

***Laptev, G. F. On a new invariant analytic method of differential geometric investigations.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 175–178. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

This is a very general outline of a method developed in the seminar of C. P. Finikov to study local imbedding properties of manifolds in homogeneous spaces with the aid of external differential forms. The principal concepts are the abstract geometrical element $G\,H$, defined as a finite continuous group G with continuous subgroup H not containing normal divisors of G, and the geometrical object of this element $G\,H$, which is a point of the space of some representation of this subgroup H. Applications are promised to the differential geometry of n-dimensional surfaces of affine space of $\frac{1}{2}n(n+3)$ dimensions and other fields.

D. J. Struik (Cambridge, Mass.).

***Petrov, A. Z. On gravitational fields.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 179–186. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

A classification of Einstein manifolds T_4, $ds^2 = g_{ij}dx^i dx^j$, $R_{ij} = kg_{ij}$, is possible by associating to every point P the local Klein space R_6 of the bivectors, with transformation group

$$\eta^{\alpha'} = A_\alpha{}^{\alpha'}\eta^\alpha, \quad A_\alpha{}^{\alpha'} = 2A_{[ij]}^{i'j'}, \quad A_i{}^{i'} = (\partial x^{i'}/\partial x^i)_P,$$
$$g_{ik}g_{jl} - g_{il}g_{jk} \to g_{\alpha\beta}, \quad \alpha, \beta, \cdots = 1, \cdots, 6.$$

If the "bivector curvature"

$$K = \frac{R_{ijkl}v^{ij}v^{kl}}{(g_{ik}g_{jl} - g_{il}g_{jk})v^{ij}v^{kl}} \to \frac{R_{\alpha\beta}v^\alpha v^\beta}{g_{\alpha\beta}v^\alpha v^\beta} \quad (v^{ij} \to v^\alpha),$$

then the elementary divisors of the matrix $\|R_{\alpha\beta} - Kg_{\alpha\beta}\|$ give at most 23 types. Of these no more than 9 types can represent real gravitational fields: 1) T_4 with real stationary curvature [(11)(11)(11)], [(1111)(11)], [(111111)]; 2) T_4 with complex stationary curvature $[11\,\overline{11}\,11]$, $[\overline{(11)(11)}\,\overline{11}]$, $[\overline{(11)(11)}\,(11)]$; 3) T_4 [(33)]. The ds^2 of the first two types is computed; to the second type belong the solutions of Schwarzschild, Kottler and Delsarte. The third type, [(111111)], is the space of constant curvature with local Minkowski metric.

D. J. Struik (Cambridge, Mass.).

Vagner, V. V. General affine and central projective geometry of a hypersurface in a central affine space and its application to the geometrical theory of Carathéodory's transformations in the calculus of variations. Trudy Sem. Vektor. Tenzor. Analizu 9, 75–145 (1952). (Russian)

This paper is closely related to the author's papers in the same Trudy 7, 65–166 (1949); 8, 144–196 (1950) [these Rev. 13, 777] and those on affine and central projective geometry of curves and surfaces developed by Dubnov [ibid. 8, 106–127 (1950); these Rev. 13, 776] and Dubnov and Skrydlov [ibid. 8, 128–143 (1950); these Rev. 13, 777]. The first sections deal with the contact of arbitrary order of m-dimensional surfaces in a central-affine E_n and its osculating hypersurfaces of given order and class. Then the influence of projective transformations in the E_n is studied. The fourth section is a discussion of the hyperquadrics of Darboux. The next sections deal with the affine and central-projective normals of a hypersurface and the general theory of hypersurfaces in a central affine E_n under transformations of the affine and central-projective group. The theory is applied to affine hyperspheres (all normals through one point) and hyperquadrics (Darboux tensor vanishes). The paper ends with the general theory of curves under the same groups.

D. J. Struik (Cambridge, Mass.).

Hiramatu, Hitosi. On affine collineations in a space of hyperplanes. Kumamoto J. Sci. Ser. A. 1, no. 1, 1–7 (1952).

By a space of hyperplanes is meant a space in which the elements are points (λ^i) together with a hyperplane element

(u_i) attached to each point. Such spaces have been the object of a recent study by K. Yano in collaboration with the author [J. Math. Soc. Japan 3, 116–136 (1951); these Rev. 13, 582]. In this paper the author studies affine collineations in such spaces, i.e., the point transformations which transform hyperplanes into hyperplanes. The investigation is carried out by using the operation of Lie derivation. Extensions are given of results obtained by various authors in previous publications, the first being one by the reviewer [J. London Math. Soc. 18, 100–107 (1943); these Rev. 5, 152]. *E. T. Davies* (Southampton).

Suguri, Tsuneo. The Gauss and Codazzi equations for a sub-space immersed in the unitary K_n-connected space. Mem. Fac. Sci. Kyūsyū Univ. A. 7, 29–34 (1952).

L'auteur envisage ici des variétés analytiques complexes C_n, de dimension complexe n, munies d'une métrique hermitique définie positive $ds^2=a_{\alpha\beta^*}dz^\alpha dz^{\beta^*}$ $(z^{\beta^*}=\bar{z}^\beta)$ et de la connexion $(\Gamma_\beta{}^\alpha, \Gamma^{\alpha^*}_{\beta^*\gamma^*})$ dont les coefficients sont donnés par $\Gamma^\alpha{}_{\beta\gamma}=a^{\sigma^*\alpha}\partial_\gamma a_{\beta\sigma^*}$ (conj.). Si C_m désigne une variété analytiquement plongée dans C_n, métrique et connexion de C_n induisent sur C_m la même structure. L'auteur forme, par la méthode de l'opérateur D, les équations de plongement de Weingarten et les équations de Gauss et Codazzi, pour une telle C_m analytiquement plongée dans C_n.

A. Lichnerowicz (Paris).

Suguri, Tsuneo. On normal coordinates in the unitary K_n-connected spaces. Mem. Fac. Sci. Kyūsyū Univ. A. 7, 35–40 (1952).

La donnée géométrique étant la même que dans le papier précédent, l'auteur se propose de rechercher dans quel cas C_n admet des coordonnées normales analytiques. Pour qu'il en soit ainsi, il faut et il suffit, si $\Lambda^\alpha{}_{\beta\gamma}$ désigne la partie symétrique de la connexion, que $\partial_{\gamma^*}\Lambda^\alpha{}_{\beta\gamma}=0$. Pour que la connexion envisagée coïncide avec la connexion riemannienne déduite de la métrique ds^2, il faut et il suffit que cette métrique soit kählerienne. Dans ce cas, l'auteur retrouve les résultats bien connus de Hodge relatifs aux coordonnées normales d'une variété kählerienne. *A. Lichnerowicz.*

Izumi, Hideo. Infinitesimal transformation in a line element space. J. Fac. Sci. Hokkaido Univ. Ser. I. 12, 1–10 (1951).

The space of line elements considered in this paper is that introduced by Kawaguchi in which the arc length of a curve is given by $s=\int[A_i x''^i+B]^{1/p}dt$ [Trans. Amer. Math. Soc. 44, 153–167 (1938)]. The structure of the space is given by two sets of functions $\Gamma^i(x, x')$ and $C^i{}_{jk}(x, x')$ so that it has properties similar to a Finsler space. The author considers infinitesimal transformations in the space, making use of the well-known Lie derivative [K. Yano, Groups of transformations in generalized spaces, Akademeia Press, Tokyo, 1949; these Rev. 10, 481]. Conditions are given in order that the space should admit infinitesimal motions. They are given in terms of the vanishing of the Lie derivatives of the curvature tensors and their covariant derivatives. *E. T. Davies.*

Tonowoka, Keinosuke. On intrinsic theories in the manifold of surface-elements of higher order. J. Fac. Sci. Hokkaido Univ. Ser. I. 12, 43–72 (1952).

The invariants associated with an integral of the form $\int_{(m)}F(x^i, \partial x^i/\partial u^\alpha)du^1\cdots du^m$, $i=1, 2, \cdots, n$; $\alpha=1, 2, \cdots, m$, have been studied extensively, for $m=n-1$, in the form of geometries based on the notion of area [E. Cartan, Les espaces métriques fondés sur la notion d'aire, Hermann, Paris, 1933]. In that case the partial derivatives $\partial x^i/\partial u^\alpha$ are replaced by the quantities p_i representing the determinant of order $n-1$ obtained by omitting one column from the matrix of the $\partial x/\partial u$. The terms considered are all assumed homogeneous of degree zero in the p_i, so that the transformations of the u-variables play hardly any part in the theory. The word tensor is used in connection with x-transformations only. This paper is concerned with the theory of quantities in which there occur the variables x as well as the partial derivatives of x with respect to u up to a certain order, and the word tensor refers to both x-transformations and u-transformations. The invariants of systems of partial differential equations with respect to both x and u transformations had already been studied by Bortolotti and in greater detail by Kawaguchi and Hombu [J. Fac. Sci. Hokkaido Imp. Univ. 6, 21–62 (1937)]. A theory which takes account of both sets of transformations is said to be intrinsic, and the author in this paper makes further contributions to the intrinsic theory, with special reference to the multiple integral in which the integrand function involves partial derivatives of order $\geqq 2$. He derives metric tensors, coefficients of affine connection, and a set of curvature tensors.

E. T. Davies (Southampton).

Deicke, Arno. Über die Finsler-Räume mit $A_i=0$. Arch. Math. 4, 45–51 (1953).

L'auteur montre que tout espace de Finsler V_n de métrique $ds=L(x, dx)$ strictement positive, et pour lequel A_i (tenseur contracté du tenseur de torsion) est nul, est un espace de Riemann (L est supposé différentiable un nombre suffisamment grand de fois). L'indicatrice est montrée être une sphère affine, puis être un ellipsoïde par généralisation d'un résultat sur les sphère affines établi par Blaschke [Vorlesungen über Differentialgeometrie, Bd. II, 1–2 Aufl., Springer, Berlin, 1923] dans le cas $n=3$.

A. Lichnerowicz (Paris).

Have

***Spain, Barry. Tensor Calculus.** Oliver and Boyd, Edinburgh and London; Interscience Publishers, Inc., New York, 1953. viii+125 pp. \$1.55.

This book is intended for the uninitiated student of tensor analysis, and gives a concise and lucid exposition of the elementary properties of Riemannian spaces. Starting from N-dimensional space (no attempt is made to give a sophisticated definition of a manifold) vectors and tensors are introduced formally, and the Riemannian metric tensor is discussed. Then follows a brief but clear exposition of covariant differentiation, geodesics, parallelism, and curvature, with applications to surfaces in 3-space. The next 20 pages are devoted to the principles of the theory of elasticity, while the last chapter shows the reader who is acquainted with the physical principles of special and general relativity how tensor methods can be used in this theory.

The book will prove to be a good introduction, both for the physicist who wishes to make applications and for the mathematician who prefers to have a short survey before taking up one of the more voluminous textbooks on differential geometry.

A. Nijenhuis (Princeton, N. J.).

NUMERICAL AND GRAPHICAL METHODS

Hammersley, J. M. Tables of complete elliptic integrals. J. Research Nat. Bur. Standards **50**, 43 (1953).

A ten-figure table of K and E with $1/k$ as argument in the interval $1.00 \leq 1/k \leq 2.00$. *L. M. Milne-Thomson.*

Luke, Yudell L., and Ufford, Dolores. A table of the complete Cicala function. J. Aeronaut. Sci. **20**, 511–512 (1953).

Tabulation to six decimals of

$$F(x) = \int_0^\infty e^{-tt}(x+t-(x^2+t^2)^{1/2})dt/xt$$

for $x=0, 0{,}01, \cdots, 0.10, 0.20, \cdots, 4.0, 4.2, 4.4, \cdots, 6.0, 6.5, 7.0, \cdots, 10.0$. *E. Reissner* (Cambridge, Mass.).

***Burkhart, Wm. H. Theorem minimization.** Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 259–263. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

Wasow, Wolfgang. Metodi probabilistici per la risoluzione numerica di alcuni problemi di analisi e di algebra. Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 336–346 (1953) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **354** (1953).

Brief description of the method with particular attention to applications to elliptic partial differential equations and to matrix inversion. *A. S. Householder.*

Wasow, Wolfgang. Random walks and the eigenvalues of elliptic difference equations. J. Research Nat. Bur. Standards **46**, 65–73 (1951).

The author treats in considerable detail the difference equation

$$(*) \qquad \Delta u + g(x, y)u = 0,$$

where Δ is the standard difference analogue of the Laplacian. The solution of (*) subject to a boundary condition is interpreted in terms of simple random walks on a lattice and it emerges as an expectation of a certain quantity related to the random walk. Conditions for the above expectation to be finite are given and there is included a discussion of random walks in an unbounded domain. Finally, a method, based on sampling, is proposed for finding the lowest eigenvalue of $\Delta u + gu$ (now Δ is an ordinary Laplacian) and the method is compared with a related one of Donsker and the reviewer [same J. **44**, 551–557 (1950); these Rev. **13**, 590]. *M. Kac* (Ithaca, N. Y.).

Wasow, W. R. A note on the inversion of matrices by random walks. Math. Tables and Other Aids to Computation **6**, 78–81 (1952).

Von Neumann and Ulam devised a Monte Carlo method for inverting certain matrices B, expanded on by the reviewer and Leibler [Math. Tables and Other Aids to Computation **4**, 127–129 (1950); **5**, 55 (1951); these Rev. **12**, 361]. Wasow [see the paper reviewed above] and (earlier) Curtiss have discussed a Monte Carlo method for solving elliptic difference equations in a domain. The author rephrases both of these in a parallel fashion. In each, the (i, j)th element of B^{-1} is estimated as the sample mean of a random variable related to the random walk of a changing mass over a set of points $P_1, \cdots, P_n$. In each method the walks start from P_i and terminate with probability 1. The von Neumann-Ulam variable G_{ij} is defined only when there is a positive probability of terminating at P_j, and is 0 unless the termination is at P_j. The Wasow variable M_{ij} is always defined, and is the total mass carried through the point P_j on the several visits to P_j in the course of a walk. The variances of G_{ij} and M_{ij} are compared for a special case. Roughly, M_{ij} has the smaller variance for all i when the probability of termination at P_j is small. *G. E. Forsythe* (Los Angeles, Calif.).

Edmundson, H. P. Monte Carlo matrix inversion and recurrent events. Math. Tables and Other Aids to Computation **7**, 18–21 (1953).

Let $p_{ik} \geq 0$ ($i, k = 1, \cdots, m$), and $\sum_{k=1}^m p_{ik} > 0$. Denote by (q^{ik}) the inverse of the matrix $(q_{ik}) = (\delta_{ik} - p_{ik})$. Two different random variables based on a certain random walk with transition probabilities p_{ik} have been defined [see Forsythe and Leibler, same journal **4**, 127–129 (1950); these Rev. **12**, 361; Wasow, the paper reviewed above] such that the expectation of these random variables is q^{ik}. The author proves that $q^{ik} - \delta_{ik} = r_{ik}q^{kk}$, where r_{ik} is the probability that a walk starting from the point i will eventually visit the point k. This relation enables him to improve a result of Wasow [loc. cit.] comparing the variances of the two random variables. *W. Wasow* (Los Angeles, Calif.).

Salzer, Herbert E. Formulas for numerical differentiation in the complex plane. J. Math. Physics **31**, 155–169 (1952).

The formulas for differentiation that are given here are based upon the approximation of $f(z)$ by the Lagrange polynomial of degree $n-1$ which assumes the n values $f(z_k)$ at certain fixed points z_k ($z_k = z_0 + kh$, where h is the length of the square in the grid, and k is a small complex integer). The points z_k are chosen to lie as close as possible in the neighborhood of z_0, e.g.:

Three-Point	Seven-Point
z_i	z_{2i}
$z_0\ z_0$	$z_i\ z_{i+1}\ z_{i+2}$
	$z_0\ z_1\ z_2$

The νth derivative of $f(z)$ can be approximated by

$$f^\nu(z) = \sum_k P_k^{(\nu)}(z) f(z_k)$$

where $P_k(z)$ is the well-known polynomial of degree $n-1$ such that $P_k(z_j) = \delta_k{}^j$ and where the summation is over the n fixed points z_k. If $z = z_0 + jh$, $P_k(z) = L_k(j)$, $P_k^{(\nu)}(z) = L_k^{(\nu)}(j)/h^\nu$, and $h^\nu f^{(\nu)}(z_j) = \sum_k L_k^{(\nu)}(j) f(z_k)$. The quantities $L_k^{(\nu)}(j)$ are expressed in the form $M_k^{(\nu)}(j)/M(\nu)$ (both $M_k^{(\nu)}(j)$ and $M(\nu)$ are integers; the $M(\nu)$ is the least common denominator of $L_k^{(\nu)}(j)$ for all values of k and j). Thus,

$$M_\nu h^\nu f_j^{(\nu)} = \sum_k M_k^{(\nu)}(j) f_k.$$

For each of the n-point configurations ($n = 3, 4, \cdots, 9$) the computation of the exact integral quantities $M(\nu)$ and $M_k^{(\nu)}(j)$ is performed by the recursion formula

$$L_k^{(\nu)}(j) = \frac{\nu}{j-k}\left[\frac{A_k}{\nu} L^{(\nu)}(j) - L_k^{(\nu-1)}(j)\right], \quad k \neq j,$$

$$L_k^{(\nu)}(j) = \frac{A_j}{\nu+1} L^{(\nu+1)}(j), \qquad k = j,$$

where $A_k=\{\prod_j'(k-j)\}^{-1}$. These formulas are obtained by ν-fold differentiation of $A_kL(j)=(j-k)L_k(j)$, by Leibnitz's theorem.

Example: Three-Point.

First derivative

$$2hf_0^{(1)}=(-2+2i)f_0+(1-i)f_1+(1-i)f_i,$$
$$2hf_1^{(1)}=(-2-2i)f_0+(3-i)f_1+(-1+i)f_i,$$
$$2hf_i^{(1)}=(2+2i)f_0+(-1+i)f_1+(-1-3i)f_i.$$

Second derivative

$$h^2f_{\text{any}}^{(2)}=-2if_0+(1+i)f_1+(-1+i)f_i.$$

S. C. van Veen (Delft).

Brun, Viggo. A generalization of the formula of Simpson for non-equidistant ordinates. Nordisk Mat. Tidskr. 1, 10–15 (1953).

Zadunaisky, Pedro E. On the numerical computation of an elliptic integral. Math. Notae 10, 1–9 (1950). (Spanish)

For α and ϕ in the neighborhood of $\pi/2$, $\phi<\alpha<\pi/2$, the value of this incomplete elliptic integral of the first kind

$$F(\alpha,\phi)=\int_0^\phi\frac{d\phi}{(1-\sin^2\alpha\sin^2\phi)^{1/2}}$$

is computed by means of the expansion

$$\int_0^\phi\frac{d\phi}{\cos\phi\left(1+\dfrac{\cos^2\alpha}{\cos^2\phi}\sin^2\phi\right)^{1/2}}$$
$$=I_1-\frac{1}{4}\frac{\cos^2\alpha}{\cos^2\phi}\sin\phi\left[1-I_1\frac{\cos^2\phi}{\sin\phi}\right]$$
$$+\frac{3}{32}\frac{\cos^4\alpha}{\cos^4\phi}\sin\phi\left[1-\frac{5}{2}\cos^2\phi+\frac{3}{2}I_1\frac{\cos^4\phi}{\sin\phi}\right],$$

where

$$I_1=\log\operatorname{tg}\left(\frac{\pi}{4}+\frac{\phi}{2}\right),\quad\text{and}\quad I_1\frac{\cos^2\phi}{\sin\phi}\to0\quad\text{for}\quad\phi\to\frac{\pi}{2}.$$

If $\alpha\leqq\phi<\pi/2$, this expansion diverges. In this case the author proposes to use Landen's transformation

$$F(\alpha,\phi)=\int_0^\phi\frac{d\phi}{(1-\sin^2\alpha\sin^2\phi)^{1/2}}$$
$$=\frac{\sin\alpha_1}{(\sin\alpha)^{1/2}}\cdot\int_0^{\phi_1}\frac{d\phi_1}{(1-\sin^2\alpha_1\sin^2\phi_1)^{1/2}}$$
$$=\frac{\sin\alpha_1}{(\sin\alpha)^{1/2}}F(\alpha_1,\phi_1),$$

where

$$\sin\alpha_1=2(\sin\alpha)^{1/2}/(1+\sin\alpha);\quad\sin(2\phi_1-\phi)=\sin\alpha\cdot\sin\phi.$$

By iteration

$$F(\alpha,\phi)=\sin\alpha_n\left(\frac{\sin\alpha_1\cdot\sin\alpha_2\cdots\sin\alpha_{n-1}}{\sin\alpha}\right)^{1/2}\cdot F(\alpha_n,\phi_n).$$

S. C. van Veen (Delft).

Ceschino, Francis. Sur une adaptation de la méthode de Graeffe au calcul automatique. C. R. Acad. Sci. Paris 236, 1945–1947 (1953).

The Brodetsky and Smeal [Proc. Cambridge Philos. Soc. 22, 83–87 (1924)] variation of the Graeffe process is the basis of this method. [See also Olver, Philos. Trans. Roy. Soc. London. Ser. A. 244, 385–415 (1952); these Rev. 14, 209.] It appears to be in part a generalization of results of J. Sebastião e Silva [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 335–343, 548–552 (1946); these Rev. 8, 354] and E. Bonneau and R. Janin [Recherche Aéronautique no. 25, 39–54 (1952); these Rev. 13, 691]. It is shown that the extraction of various 2^mth roots can be avoided by the solution of a certain system of linear equations. *J. Todd* (Washington, D. C.).

Kimball, Bradford F. Note on computation of orthogonal predictors. Ann. Math. Statistics 24, 299–303 (1953).

The least-squares problem is that of obtaining the orthogonal projection Xh of a known vector y upon the column space of the known matrix X of linearly independent columns, and hence of finding h such that $Xh=y+e$, $X^Te=0$. The columns of X can be orthogonalized by forming $\Phi=XV^T$, where V^T is upper triangular and $\Phi^T\Phi=I$. Then $h=V^Tt$ where $t=\Phi^Ty$. The main result is stated as a lemma (to be applied in a subsequent paper) to the effect that the adjunction of any additional columns to X, and hence of additional elements to t, leaves unaffected those elements of t already found. The matrix V is also related to the matrices obtained in the Gauss-Doolittle-Choleski process of solving $X^TXh=X^Ty$, and the matrix relations are interpreted element-wise. *A. S. Householder*.

***Rubinstein, H., and Rutledge, J. D. High order matrix computations on the Univac.** Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 181–186. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

***Orden, Alex. Solution of systems of linear inequalities on a digital computer.** Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 91–95. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

Shortley, George. Use of Tschebyscheff-polynomial operators in the numerical solution of boundary-value problems. J. Appl. Phys. 24, 392–396 (1953).

Um die Poissonsche Differentialgleichung bei vorgegebenen Randwerten näherungsweise zu lösen wird zur Differenzengleichung übergegangen. Sei

$$\omega u_{ij}+\tfrac{1}{4}K^2\alpha_{ij}=u_{ij}$$

die aufzulösende Differenzengleichung wobei der Operator ω durch

$$\omega u=\tfrac{1}{4}(u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1})$$

erklärte sei. Sei ferner O der durch

$$Ou_{ij}=\omega u_{ij}+\tfrac{1}{4}K^2\alpha_{ij}$$

erklärte lineare inhomogene Operator. Dann läuft die Iterationsmethode darauf hinaus, dass man O^n auf eine zweckmässig gewählte Näherungsfunktion anwendet. Dem Ergebnis einer früheren Arbeit [Flanders und Shortley, J. Appl. Phys. 21, 1326–1332 (1950); diese Rev. 12, 640] zufolge erscheint es zweckmässiger an Stelle von O^m den durch $\varphi_m(O)=T_m(O/\lambda_1)/T_m(1/\lambda_1)$ erklärten Operator zu benützen. Dabei ist T_m das Tschebyscheffsche Polynom $m$ter Ordnung und λ_1 der grösste zu ω gehörige Eigenwert (bez. ein passend geschätzter Näherungswert). Ziel der vorliegenden Arbeit ist es eine Schätzung für die erzielte Ersparnis an Rechenarbeit bei einer grossen Anzahl N von Gitterpunkten zu finden. Als Mass für die Genauigkeit wird

der Wert des Verkleinerungsfaktors beim Koeffizienten des ersten Gliedes in dem nach Eigenfunktionen des Operators w entwickelten Fehler nach Anwendung von O^n bez. von φ_m betrachtet.

Um die Grössenordnung des ersten Eigenwerts zu schätzen betrachtet der Verfasser den Fall eines quadratischen Bereichs, wo das Eigenwertproblem elementar lösbar ist. Abschätzung von n bez. m bei vorgegebener Genauigkeit in angegebenen Sinn ergibt, dass die Ersparnis von Rechenarbeit proportional mit $N^{1/2}$ ist. Die Vorteile der Rechnung mit dem Operator φ_m wird auch an einem numerischen Beispiel dargetan. Es werden auch Verbesserungsvorschläge und Verallgemeinerungen im Anschluss an die erwähnte Arbeit besprochen. *P. Funk* (Wien).

✶Alt, Franz L. Boundary value problems for multiply connected domains. Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 193–195. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

✶Bergman, Stefan. The solution of boundary value problems by the method of the kernel function. Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 187–192. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

Koehler, Fulton. Estimates for the errors in the Rayleigh-Ritz method. Pacific J. Math. **3**, 153–164 (1953).

Die Arbeit behandelt Abschätzungen bei einem n-gliedrigen Ansatz entsprechend der Methode von Rayleigh-Ritz für das Eigenwertproblem $Ly=\lambda y$, mit vorgegebenen linearen Randbedingungen, wobei die Randbedingungen den Parameter nicht enthalten. Dabei ist der $k$te Eigenwert durch die Minimumsforderung $\lambda_k=\min\,(\phi, L\phi)/(\phi,\phi)$ unter Berücksichtigung der zugehörigen Orthogonalitätsbedingungen, bzw. durch die entsprechende, bekannte Maximum-Minimum Forderung, gekennzeichnet. Ausgangspunkt für die Abschätzungen ist die unmittelbar aus der Schwarz'schen Ungleichung sich ergebende Ungleichung

$$\frac{(\phi, L\phi)^2}{(\phi,\phi)^2} \leq \frac{(L\phi, L\phi)}{(\phi,\phi)},$$

verbunden mit der Kennzeichnung des Eigenwertes durch das der rechten Seite der obigen Ungleichung entsprechende Variationsproblem. Die Fehlerschätzung wird in Beziehung gebracht mit einem Ausdruck von der Form

$$\epsilon(n)=\left[\int\int (K-A_n)^2 dPdQ\right]^{1/2}$$

wobei A_n den in den Koordinatenfunktionen nach der Methode der kleinsten Fehlerquadrate gebildeten Näherungsausdruck bedeutet. Auf diese Weise gelingt es nicht nur Schätzungen für die Güte der Annäherung an die Eigenwerte, sondern auch an die Eigenfunktionen anzugeben. *P. Funk* (Wien).

Svirskiĭ, I. V. On the exactness of variational methods for the determination of the critical forces for longitudinal bending. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk **2**, 63–76 (1950). (Russian)

The author outlines a method for the estimation of the error committed in employing variational methods such as Rayleigh-Ritz' and Galerkin's for approximating eigenvalues of elastic systems. In particular, it is shown that the method may be effectively applied to the problem of longitudinal bending of beams of non-constant cross-section. *J. B. Diaz* (College Park, Md.).

Woods, L. C. The relaxation treatment of singular points in Poisson's equation. Quart. J. Mech. Appl. Math. **6**, 163–185 (1953).

"If Φ is harmonic or is a solution to Poisson's equation, it may have singular points in the field or on the boundary at which it (a) has finite values, but has infinite derivatives, (b) has logarithmic infinities, or (c) has simple discontinuities. This paper describes methods that can be adopted by computers using the relaxation technique when working in the neighborhood of these points. Methods of obtaining accurate derivatives and integrals in these neighborhoods are also given. Four examples illustrate the methods and suggest that the accuracy is comparable with that obtained in similar problems without singularities."

The principal idea used is that the above singularities of harmonic functions of two independent variables can be explicitly approximated by elementary functions of a complex variable, $f(re^{i\theta})$. The principal advantage of the present method is that such information is used to modify the formula for residuals at only the net points nearest to a singularity. The customary residual scheme is used for the rest of the net. (During the iterations of the relaxation process it is necessary to revaluate certain parameters which may appear in the few modified residual formulas.) The article contains much good, practical advice. An extension of the technique to treat a three-dimensional problem with a simple singularity is given. *E. Isaacson.*

Plunkett, R. On the rate of convergence of relaxation methods. Quart. Appl. Math. **10**, 263–266 (1952).

In general the finite approximation to partial differential equations may be written $Az+f=0$ where z is an n-element column matrix of the unknown values, and n is the number of points taken in the region. A is an $n\times n$ square matrix, with a high degree of regularity; the elements of the main diagonal are the largest and most of the other elements are zero. f is a known column matrix. If z_m is the mth approximation to z, the matrix of the residuals is defined by: $Az_m+f=R_m$. Let the elements of R_m be random variables with a mean value zero, a maximum value ρ_m, and a standard deviation $b\rho_m$, where b is a number less than one. Then the mathematical expectation of $|R_m|$ is: $|R_m|=n^{1/2}b\rho_m$. It is shown that for large values of n

$$\text{(1)}\qquad \frac{|R_{m+n}|}{|R_m|}\sim e^{-k}$$

where $k=(1-\frac{1}{2}l)/b^2$ (l is a small number greater than one; for Poisson's equation $l=1.25$, for the biharmonic $l=1.72$). k is a number not much greater than one unless b, the standard deviation ratio, is very small.

In general agreement with the usual experience in relaxation techniques, something like 20 steps as are indicated in (1) will reduce the error as measured by $|R_m|$ to at most 10^{-6} of its original value. Comparison with the best results obtained by Frankel for iteration methods for Poisson's equation and for the biharmonic equation [Math. Tables and Other Aids to Computation **4**, 65–75 (1950); these Rev. **13**, 692] leads to the conclusion that there is no saving in a relaxation method, which is actually more difficult to program, unless the problem is more complicated than the biharmonic equation. *S. C. van Veen* (Delft).

Allen, D. N. de G., and Dennis, S. C. R. The application of relaxation methods to the solution of differential equations in three dimensions. II. Potential flow round aerofoils. Quart. J. Mech. Appl. Math. **6**, 81–100 (1 plate) (1953).

[For part I see same J. **4**, 199–208 (1951); these Rev. **13**, 165.] No stream-function is available for three-dimensional problems of potential flow of incompressible fluid around an aerofoil; the authors therefore devise a relaxation technique, based on a cubical lattice, which gives the distribution of velocity potential for such problems. The method involves finding the position of an initially unknown boundary, the stream-surface from the trailing edge. It is applied to the two-dimensional flow past an inclined flat plate, as a preliminary example. It is then applied to the flow past a plane aerofoil of rectangular profile and a plane swept-back aerofoil; the results for the variation of the circulation and the position of the local aerodynamic centre along the span are compared with some calculations by Falkner [Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda nos. **2596** (1952) and **2740** (1952)] and show close agreement. The paper ends with a discussion of the grading from a region of given cube size to another where the cubes have sides one-half as long. *D. C. Pack* (Manchester).

Myard, Francis. Résolution grapho-mécanique des relations

$$\varphi_{(z)}=f_{(z)}\cdot f'_{(z)},\quad \psi_{(z)}=f_{(z)}/f'_{(z)}.$$

C. R. Acad. Sci. Paris **236**, 2143–2145 (1953).

Vil'ner, I. A. The analytic theory of nomographing a function of a complex variable of the first class. Mat. Sbornik N.S. **27**(69), 3–46 (1950). (Russian)

In various earlier papers [see especially Doklady Akad. Nauk SSSR (N.S.) **58**, 729–732 (1947); these Rev. **9**, 534] the author has given summaries and certain practical applications of his work on representing an analytic function $p+iq=f(a+ib)$ as an alignment chart. The present paper gives a more detailed account of this work, stressing its mathematical aspects. The paper is, however, far from self-contained; essential steps in arguments and even statements of results are abbreviated by detailed references to the author's dissertation (1946) which is unfortunately not available to the reviewer. As not explicitly mentioned in previously reviewed papers the following may be mentioned: (1) several further sets of necessary and sufficient conditions for $w=f(z)$ to be of first class (i.e., no more than two of the four scales curved), (2) the theorem and some consequences that if $z=\int^w f(\zeta)d\zeta$ is of first class with z scales straight, the same is true of $z=\ln f(w)$, and (3) the derivation of relations among nomograms for branches of a multiple-valued function if they are of the first class.
R. Church (Monterey, Calif.).

✶Couffignal, Louis. Les machines à penser. Les Editions de Minuit, Paris, 1952. 158 pp. (4 plates).

This book is the third in a series entitled "L'homme et la machine". It is concerned primarily with computing machines and with some of the analogies that exist between them and the human nervous system. The author considers both digital and analogue machines and gives a brief description of the historical background of the subject and of some aspects of the ideas underlying these machines. He has evidently written on such a level that the work should be understood by an intelligent layman. The book broadly covers the following topics: digital computing machines, analogue machines, analogies between the nervous system and a computational machine, and a discussion of a mechanization of logic.

Three more volumes of this series are in preparation. The collection should form a valuable account of the impact of machines of all sorts on our modern society.
H. H. Goldstine (Princeton, N. J.).

Willers, Friedrich-Adolf. Mathematische Maschinen. Wissensch. Ann. **2**, 280–298 (1953). Expository paper.

Ramsayer, K. Die erste Kleinfunktionsrechenmaschine. Z. Angew. Math. Mech. **33**, 215–216 (1953).

Bösch, Walter. Rechnen mit komplexen Zahlen auf einer mechanischen Multipliziermaschine. Z. Angew. Math. Physik **4**, 214 (1953).

✶Chase, George C. History of mechanical computing machinery. Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 1–28. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

✶Gordon, B. M., and Nicola, R. N. Special-purpose digital data-processing computers. Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 33–45. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

✶Auerbach, Albert. The Elecom 100 general purpose computer. Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 47–51. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

✶Astrahan, M. M., and Rochester, N. The logical organization of the new IBM scientific calculator. Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 79–83. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

✶Wallace, Richard A. The maze solving computer. Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 119–125. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

✶Levin, Joseph H. Construction and use of subroutines for the SEAC. Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 173–180. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

Grant, Fraser S. Three dimensional interpretation of gravitational anomalies. II. Geophysics **17**, 756–789 (1952).

This is the second paper by the author dealing with the geological interpretation of gravimetric surveys. The method as described in the first paper [Geophysics **17**, 344–364 (1952); these Rev. **14**, 211] is based on the use of multiple moments of the observed and mapped quantity. These moments are to be computed from the smoothed data and just for a single point on the map, namely, above the projection on the map's plane of the center of gravity of disturbing masses which create the anomaly.

The use of multiple moments presupposes a map of pure residual anomaly, that is, a map of a single tectonic phenomenon. It necessitates, therefore, not only the elimination of regional anomaly, but also a complete separation of the anomaly to be interpreted from the other residual over-

lapping anomalies which distort its gravimetric picture. In general, such a separation is very difficult to achieve, but granted it, the method in itself is an excellent one. All the necessary details such as smoothing the data (elimination of regional anomaly), charts, and templates for computing the numerical values of multiple moments, as well as the tables of all functions required in the practical application of the method, are provided.

Two useful examples are given at the end. The first one describes the solution of an artificial interpretation problem. The second one gives the interpretation of an actual survey conducted over a sulphide body and confirmed by extensive drilling. It seems that both papers on the method of multiple moments represent a valuable contribution to applied geophysics. To understand their theory requires a sufficient knowledge of higher mathematics. *E. Kogbetliantz.*

RELATIVITY

***Jordan, Pascual. Schwerkraft und Weltall. Grundlagen der theoretischen Kosmologie.** Friedr. Vieweg & Sohn, Braunschweig, 1952. viii+207 pp. DM 15.80.

Jordan's announced purpose in writing this book is twofold: to give a logically complete exposition of Einstein's general theory of relativity, and to present a unified account of his own and related work on the projective relativistic theory of cosmology. In both of these aims he has succeeded admirably.

The first chapter, of fifty pages, is devoted to a presentation of Riemannian geometry and its natural mathematical tool, the Ricci or absolute differential calculus. The geometrical topics are selected with an eye to their application to the cosmological problem, without, however, distorting the principal aim to develop the geometrical framework for the general theory of relativity and its projective extension. Jordan here makes a point of avoiding the tedious computations traditionally associated with the derivation of the curvature tensor and the algebraic and differential identities which it satisfies. Although it is my belief that the elegant tricks utilized to avoid these computations will not in fact do away with the conscientious student's sweating through great gobs of tensor analysis in the traditional component form, the student will be able to turn to the treatment here given with a greater understanding of the meaning of the whole tensor.

The second chapter, of about the same length, offers a concise account of Einstein's theory of gravitation, with special emphasis on the Schwarzschild field and on cosmology, under the explicit rejection of the "cosmological constant". While the critical expert may raise objections at some points—as at the assertion (p. 62) that the linear element (13), which contains superfluous second order terms in the planetary application, is the simplest Newtonian approximation, or at the naive interpretation (p. 70) of the significance of the Schwarzschild "singularity" at $r=2m$ —the account is on the whole quite sound. Excellent features are the inclusion (§18) of an account of the role of causality in general relativity, based on K. Stellmacher's mathematical analysis, and the discussion (§19) of variation principles.

Chapter III, on projective relativity, has been included for logical completeness, as Jordan finds in this theory the appropriate background for his own solution of the cosmological problem. The reader who is prone to shy away from mathematical constructions with little direct contact with physics may well heed the author's warning to skip this chapter, although it does give what is perhaps the most convincing argument for the contention that the gravitational "constant" should in fact be treated as a scalar field quantity, varying from region to region of space-time.

The crux of the book, Jordan's own cosmological theory, is contained in the final Chapter IV of some seventy pages and is given in considerable detail. The serious student of the cosmological problem will do well to spend the time necessary to come to an understanding of Jordan's developments. Here he will find points to challenge his imagination and ingenuity, as well as points on which his flights of fancy will indubitably boggle at carrying him to the dizzy heights attained by Jordan! But it would be unfair or unwise to dispose offhand of the more startling of his conclusions—even the singular appearance of matter (pp. 174, 185) in supernovae which spring Minerva-like full-panoplied from the depths of Nowhere, or the revolutionary explanation (p. 198) of Wegener's continental drift hypothesis in terms of the secular decrease of the constant of gravitation! These and others are suggestive points which should be borne in mind when attempting to resolve the puzzles now confronting cosmology. Jordan's effort is indeed a welcome complement to the excellent accounts of the relativistic and a prioristic cosmologies recently given by Bondi and by McCrae, which pass somewhat cavalierly over the present more radical field theory. *H. P. Robertson.*

Matte, Alphonse. Sur de nouvelles solutions oscillatoires des équations de la gravitation. Canadian J. Math. 5, 1–16 (1953).

Dans un espace-temps extérieur de la relativité générale, rapporté à des coordonnées locales où les lignes de temps ($t=x^0$) sont orthogonales aux sections d'espace

$$ds^2=c^2dt^2-a_{rs}dx^r dx^s \quad (r,\ s,\ \text{tout indice latin}=1,\ 2,\ 3)$$

l'auteur introduit les deux tenseurs d'espace E et H définis par

$$4E^{rs}=R_{uv,\,mn}\epsilon^{uvr}\epsilon^{mns}, \quad 2cH_r{}^s=R_{r0,\,uv}\epsilon^{uvs}$$

où ϵ désigne ici le tenseur-indicateur de Kronecker relatif à une section d'espace. Compte tenu des équations d'Einstein $R_{\mu\nu}=0$ (μ, ν, tout indice grec $=0, 1, 2, 3$) les tenseurs E^{rs} et H^{rs} sont symétriques et de traces nulles. Inversement toutes les composantes du tenseur de courbure de l'espace-temps s'expriment aisément à l'aide de E, H, ϵ et des potentiels de gravitation. De plus $E_{rs}E^{rs}-H_{rs}H^{rs}$ et $E_{rs}H^{rs}$ sont des invariants d'espace-temps égaux, respectivement, à des facteurs constants près, au carré du tenseur de courbure et à $\epsilon_{\mu\nu\rho\sigma}R^{\alpha\beta\mu\nu}R_{\alpha\beta}{}^{\rho\sigma}$.

L'auteur suppose ensuite que les potentiels de gravitation oscillent localement avec une fréquence n, l'amplitude d'oscillation étant de l'ordre de n^{-r} ($r\geqq 1$). Pour n suffisamment grand, les équations $\nabla_\alpha R^\alpha{}_{\beta,\gamma\delta}=0$, conséquence classique des équations d'Einstein, conduisent pour E et H aux équations approchées:

$$\epsilon^{uvs}\overline{\overline{\nabla}}_v E_{ru}+\frac{1}{c}\frac{\partial H_r{}^s}{\partial t}\sim 0, \quad \epsilon^{uvs}\overline{\overline{\nabla}}_v H_{ru}-\frac{1}{c}\frac{\partial E_r{}^s}{\partial t}\sim 0,$$

$$\overline{\overline{\nabla}}_s E^{rs}\sim 0, \qquad \overline{\overline{\nabla}}_s H^{rs}\sim 0,$$

où $\overline{\overline{\nabla}}$ est l'opérateur de dérivation covariante dans la métrique d'espace. Ces relations présentent avec les équations

de Maxwell une analogie très étroite. Les interprétations physiques possibles de cette intéressante étude ne sont pas données. *A. Lichnerowicz* (Paris).

Kaempffer, F. A. The physical meaning of auxiliary conditions in the theory of gravitational waves. Canadian J. Physics 31, 501–503 (1953).

The coordinate conditions are examined which are necessary to reduce the gravitational field-equations in first approximation (that is, for weak fields) to a set of ordinary wave equations. These conditions are evaluated for the case of the gravitational field produced by an accelerated particle.

The author claims that these conditions restrict the consideration to some observers, whereas the reviewer believes that they restrict the motion of the particle to a uniform one. *L. Infeld* (Warsaw).

Maravall Casesnoves, Dario. Dynamical solution of the problem of a single body in the theory of relativity. Hypothesis on the origin of cosmic rays. Euclides, Madrid 13, 62–72 (1953). (Spanish)

The author obtains, by devious means, a line element which he maintains represents the gravitational field of a spherically symmetric body which is emitting radiation at the expense of its mass. Comparing the resulting expression for the flux of energy with the rate of increase of mass in an expanding universe, of a type described in previous articles of the series [Euclides, Madrid 10, 427–432 (1950); 11, 205–210, 391–404 (1951); 12, 140–151 (1952)], the author finds support for his contention that cosmic rays are the materialization of the energy dealt with in his proposed solution of the one-body problem. *H. P. Robertson.*

Sailer, Herbert. Die zehn allgemeinen Integrale der Bewegungsgleichungen in der Mechanik der speziellen Relativitätstheorie. Acta Physica Austriaca 7, 155–163 (1953).

A system of n particles moving under their internal interactions according to the mechanics of special relativity is considered. It is shown that, as in classical mechanics, the motion admits of ten integrals, viz., three integrals expressing the conservation of linear momentum, the energy integral, three integrals expressing the conservation of angular momentum, and three integrals which are the analogues of the classical formulae stating that the center of mass moves in a straight line with constant speed. *G. C. McVittie* (Urbana, Ill.).

von Krbek. Grundzüge der speziellen Relativitätstheorie. Wissensch. Z. Univ. Greifswald. Math.-Nat. Reihe 1, no. 2, 32–38 (1952).

Expository paper.

Ueno, Yoshio. On the equivalency for observers in the special theory of relativity. Progress Theoret. Physics 9, 74–84 (1953).

Derivation of the most general group of linear transformations between relativistically equivalent observers. [Cf. W. Pauli, Jr., Encykl. Math. Wiss., Bd. V, 2. Teil, Teubner, Leipzig, 1921, p. 556 for result and references to the previous work of Ignatowsky (1910–11) and Frank and Rothe (1911–1912)]. *H. P. Robertson.*

★Garcia, Godofredo. The new theory of general relativity. Have Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 139–160. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

L'auteur donne un exposé d'ensemble de sa théorie relativiste de la gravitation [Actas Acad. Ci. Lima 14, no. 1, 3–41, no. 2–3–4, 3–6, 7–8, 9–14, 15–19, 20–27 (1951); ces Rev. 12, 866; 13, 696, 597], qui est en connexion étroite avec les idées de G. D. Birkhoff (théorie alternative); cette théorie diffère de celle de Birkhoff par modification de la métrique et conservation dans celle-ci d'un potentiel gravitationnel g_{00}. Dans le présent papier, on trouve notamment le calcul du potentiel g_{00}, qui conduit à des effets égaux aux effets einsteiniens, l'étude de la déviation des rayons lumineux, le calcul des forces perturbatrices par rapport à la loi de Newton et la comparaison avec le potentiel de Levi-Cività en seconde approximation, ainsi que des éléments de mécanique des milieux continus. *A. Lichnerowicz* (Paris).

MECHANICS

McKinsey, J. C. C., Sugar, A. C., and Suppes, Patrick. Axiomatic foundations of classical particle mechanics. J. Rational Mech. Anal. 2, 253–272 (1953).

The authors try to give an axiomatisation of classical particle mechanics. The system of the authors, which expresses their view of classical mechanics, is based on the following primitive notions: a set P the elements of which are called particles, a connected real interval T, an n-dimensional vector space V, a scalar function $m(p)$, $p \in P$, and two vector functions $s(p, t)$ and $f(p, t, i)$, $p \in P$, $t \in T$, $i \in I$ (set of positive integers). The system (P, T, m, s, f) is called a system of particle mechanics if the following axioms are satisfied: (1) P is a nonempty finite set; (2) $d^2s(p, t)/dt^2$ exists; (3) $m(p)>0$; (4) $\sum_i f(p, t, i)$ is absolutely convergent; (5) $m(p)d^2s(p, t)/dt^2=\sum_i f(p, t, i)$. From these axioms a theorem is derived corresponding to Newton's first law and it is shown that P, T, m, and f together with appropriate initial conditions determine s. Special systems are defined: Newtonian systems and ultra-classical systems, both closely connected with the third law. It is shown that every system of particle mechanics is a subsystem of a Newtonian system. The primitive notions m, s, and f are mutually independent; it is shown that none of these three functions is uniquely determined by the two other functions. *J. Haantjes* (Leiden).

McKinsey, J. C. C., and Suppes, Patrick. Transformations of systems of classical particle mechanics. J. Rational Mech. Anal. 2, 273–289 (1953).

This paper is devoted to a determination of the class of transformations which take arbitrary systems of particle mechanics [see the preceding review] into systems of particle mechanics. Let (P, T, m, s, f) be such a system, α, β, $\gamma>0$ real numbers, A a nonsingular n-matrix, and p and q two n-dimensional vectors. Then the transformations

$$(1)\quad \begin{gathered} t'=\alpha+\beta t;\ m'(p)=\gamma m(p);\ f'(p, t', i)=\gamma f\left(p, \frac{t'-\alpha}{\beta}, i\right)\cdot A, \\ s'(p, t')=\beta^2 s\left(p, \frac{t'-\alpha}{\beta}\right)\cdot A+t'p+q \end{gathered}$$

define a new system (P, T', m', s', f'). It is shown that in a certain large class of transformations (1) are the only transformations which take every system of particles into such a system. Newtonian systems are transformed into Newtonian systems but every ultra-classical system is carried over into an ultra-classical one if and only if A is a similarity matrix. *J. Haantjes* (Leiden).

von Krbek. Grundzüge der Mechanik. Wissensch. Z. Univ. Greifswald. Math.-Nat. Reihe **1**, no. 2, 22–31 (1952).

Expository paper.

✶Grodzinski, P. Getriebelehre. I. Geometrische Grundlagen. 2te Aufl. Sammlung Göschen, Bd. 1061. Walter de Gruyter & Co., Berlin, 1953. 159 pp. DM 2.40.

There is a remarkable wealth of material in this little book limited to plane mechanisms. The topics are: geometry of motion, four-bar linkages, cams, and elements of spur-gearing. A reader fond of concise presentation, and in need of a quick introduction to the field, will like the booklet. A few recent developments have been noted in this second edition. *A. W. Wundheiler* (Chicago, Ill.).

✶Wundheiler, Alexander W. Some formulas for bar linkages. Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 19–22. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author considers plane bar linkages with not more than two turning pairs on each bar; ϕ_s is the angle of the bar B_s and the fixed bar. He introduces certain functions M_k^s and N_k^s of the bar directions ϕ_k, defined by the linkage structure alone; they are quotients of determinants, whose terms are cos φ_k, sin ϕ_k, or zero. The angular corrections $\Delta\phi_k$ are bilinear in the M's and N's. Assuming the rotation of the input crank to be uniform at unit speed the angular velocities $\dot\phi_k$ are given as quotients of expressions bilinear in the M's, N's, cos t and sin t. Thus a differential equation is obtained for all the functions generated by the rotation of the bars, for a given linkage structure. *O. Bottema.*

Gran Olsson, R. Bemerkungen zur ebenen Bewegung von Rotationskörpern. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 32–37 (1953).

Elementare Betrachtungen über das reine Rollen eines Rotationskörpers auf einer schiefen Ebene. *O. Bottema.*

Tzénoff, Iv. Détermination de la translation et la rotation d'un corps solide, étant données les vitesses de trois de ses points. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie II. **47**, 59–66 (1952). (Bulgarian. French summary)

Nous avons déterminé la translation $\mathbf{v}_0$ et la rotation ω d'un corps solide, lorsqu'on connaît les vitesses de trois de ses points, en fonction des coordonnées vectorielles des trois points et en fonction de leurs vitesses.

Résumé de l'auteur.

Agostinelli, Cataldo. Sul moto di rotolamento su un piano orizzontale di una sfera pesante a struttura giroscopica rispetto a un diametro. Rivista Mat. Univ. Parma **3**, 327–338 (1952).

The author considers a rigid body, in the form of a sphere, rolling without slipping on a horizontal plane. It is assumed that the centroid is at the center of the sphere, but not that the body is dynamically symmetrical about the center. It is shown that the scalar product of the angular velocity vector and a certain fixed vector c is constant. It is then shown that the motion of the body can be determined by quadratures in each of the following cases: (1) c is perpendicular to the horizontal plane; (2) two of the principal moments of inertia of the body are equal. Some interesting special cases are discussed in detail. *L. A. MacColl* (New York, N. Y.).

Murden, William P., Jr. The motion of a rigid body with nonholonomic constraint. Texas J. Sci. **5**, 192–197 (1953).

Sretenskiĭ, L. N. Motion of the Goryačev-Čaplygin gyroscope. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1953**, 109–119 (1953). (Russian)

The problem of motion under gravity of a body one of whose points 0 is fixed has been solved by Euler (when the fixed point 0 is the center of gravity of the body), Lagrange (when two of the principal moments of inertia at the point 0 are equal, i.e., $A=B$, and the center of gravity lies on the third axis of inertia of the body), and S. Kovalevskaya (when $A=B=2C$ and when further the center of gravity is situated in the plane of the equal moments of inertia). In each of these cases the general solution contains five arbitrary constants of integration. Goryačev [Mat. Sbornik **21**, 431–438 (1900)] showed that the problem is also solvable when $A=B=4C$, the center of gravity is in the plane of the equal moments of inertia at 0, and when further the angular momentum about the vertical through 0 is zero. In this case the solution contains three arbitrary constants. Finally, Čaplygin [Coll. Works, vol. 1, Gostehizdat, Moscow-Leningrad, 1948, pp. 118–124; these Rev. **14**, 609] showed that under the assumptions made by Goryačev another particular integral exists, and hence there exists a solution depending upon four arbitrary constants.

Let the line through the fixed point O and the center of gravity be taken as the Ox-axis, and let the center of gravity be at the distance α from O; let the Eulerian angles ϵ, ω, φ which define the position of the principal axes of inertia $Oxyz$ with reference to fixed rectangular axes $OXYZ$ of which the axis OZ is vertical be defined as follows: ϵ is the angle between the axes OZ and Ox, ω is the angle between the OX-axis and the line of intersection $O\gamma$ of the planes XOY and yOz, and φ is the angle between $O\gamma$ and the Oy-axis. Further let p, q, r be the components along the axes $Oxyz$ of the angular velocity of the body, and let P be its weight.

The author investigates the motion of the Goryačev-Čaplygin gyroscope assuming that initially: (i) the axes OX and Ox coincide while the axes Oy and Oz make with the axes OY and OZ respectively an angle θ_0, and (ii) that a large spin is given about the Ox-axis to the body, i.e., $p=p_0$, $q=r=0$, where p_0 is large. The results obtained may be summed up as follows. The amplitudes of the oscillations of cos ϵ vary in such a way as to produce beats. The period of the beats is $4\pi p_0/3a$, while the period of the small oscillations, constituting the beats, is π/p_0, where $a=P\alpha/C$. The axis of the gyroscope, performing the above mentioned oscillations, passes through the equatorial plane of the fixed sphere of radius one described about O at times

$$t_n=\frac{4p_0}{3a}(n\pi-\lambda\sin\theta_0),\quad t_m=\frac{1}{p_0}(m\pi-\theta_0),$$

where $\lambda=a/2p_0^2$ and $n, m=0, \pm1, \pm2, \cdots$.

At each of these instants the angle ω changes the sense of its variation, i.e., it passes from an increasing angle to a decreasing one and conversely. In addition ω attains its relative extremum values at times $t_n'=t_n+4\pi p_0/6a$. The angle φ varies almost proportionally to the time with velocity p_0 near the instants t_n when the axis of the gyroscope is horizontal. Near the instants of time corresponding to maximum inclinations of the axis of the gyroscope to the equator of the fixed sphere the angle φ changes the sense of its variation. *E. Leimanis* (Vancouver, B. C.).

Košlyakov, V. N. On certain particular cases of integration of the dynamical equations of Euler connected with the motion of a gyroscope in a resisting medium. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 137–148 (1953). (Russian)

Suppose that the equatorial moments of inertia A and B of a gyroscope differ only slightly from each other, that $B>A$ and that the axial moment $C>A$. Further assume that the components of the moment generated by the resistance forces are $-\lambda Ap$, $-\lambda Bq$, $-\lambda Cr$, where λ is some coefficient of proportionality and p, q, r are the components of the angular velocity of the flywheel. A solution of the Eulerian equations is sought in the form of power series in terms of the small parameter $\epsilon=(B-A)/C$. The method of successive approximations is applied in such a way that the zero approximation corresponds to the symmetric case, when $A=B$.

In the case of a symmetric gyroscope ($A=B$) and under the assumption that the moment of the resisting forces is of certain particular forms, the Eulerian equations are integrable in terms of Bessel functions and a degenerate hypergeometric function. Furthermore, it is shown that, if the angular velocity of the flywheel decreases in such a way that the moment of the resisting forces can be considered as varying proportionally to the first or to the second power of the angular velocity, the equation, determining the angle of rotation of the inner Cardan ring, can be integrated in terms of Bessel functions of zero order or cylindrical functions respectively [cf. also Košlyakov, Akad. Nauk SSSR. Inženernyĭ Sbornik 6, 185–196 (1950)]. *E. Leimanis.*

Kosmodem'yanskiĭ, A. A. General theorems of the dynamics of a body of variable mass. Moskov. Gos. Univ. Učenye Zapiski **152**, Mehanika 3, 13–42 (1951). (Russian)

This paper covers the same ground as Lectures 8 and 9 of the paper reviewed below. *R. A. Rankin.*

Kosmodem'yanskiĭ, A. A. Lectures on the mechanics of bodies of variable mass. Moskov. Gos. Univ. Učenye Zapiski **154**, Mehanika 4, 73–180 (1951). (Russian)

These lectures are from a course which the author has given since the session 1943/44 at the Soviet academy of military and aeronautical engineering. The paper begins with a historical account of theoretical mechanics giving especial emphasis to the work of Meščerskiĭ (1859–1935) and Tsiolkovskiĭ (1857–1935) who evidently were the first to consider mathematically problems concerning bodies of variable mass. The work of Esnault-Pelterie, Goddard, Oberth, Levi-Cività, Tsander and Kondratjuk is mentioned, but there is no reference to more recent work done outside Russia.

First lecture. Here are derived the equations of motion of a particle (the author uses the word 'point' for the word 'particle' which he uses in a different connexion) of variable mass, including the equation

$$M\frac{d\mathbf{v}}{dt}=\mathbf{F}+\mathbf{V}_r\frac{dM}{dt}, \tag{1}$$

where M is the mass of the particle, $\mathbf{v}$ its velocity, t the time, $\mathbf{F}$ the external force acting and $\mathbf{V}_r$ the relative velocity of the ejected part. In the case when the absolute velocity of the ejected part is zero this equation was first obtained by Meščerskiĭ [Thesis, St. Petersburg, 1897]. Equation (1) is resolved into three scalar equations for the case when $\mathbf{F}$ is composed of gravity and air resistance in the direction opposing motion. When these forces are neglected, the now well-known law $v=v_0+V_r\log(M_0/M)$ is deduced from (1) and attributed to Tsiolkovskiĭ (1903). The 'burning laws' (i) $f(t)=M/M_0=1-\alpha t$, (ii) $f(t)=e^{-\alpha t}$ are investigated in detail.

Second lecture. The vertical movement of a particle of variable mass with the second burning law, under a uniform field of gravity and without air resistance, is considered; the maximum height H attainable, for given α, V_r and final mass M_e, is obtained. It is deduced that H is greatest when $\alpha=\infty$, i.e., when the mass is reduced from M_0 to M_e instantaneously. The maximum height attainable at the instant of 'all-burnt' is also obtained. Rectilinear motion without gravity with the first burning law and a resistance proportional to the square of the velocity is also considered.

Third lecture. In §1 the motion under a resistance of the form $a+bv$ is considered for the first burning law. In §2 the author considers a jet aircraft moving horizontally against air resistance so that the lift just compensates the weight. The application of the particle theory to this problem is, of course, only possible as an approximation. The distance traveled during the period when the jet is acting is calculated, for the first burning law, and the optimum value of α is obtained. In §3 vertical motion under uniform gravity subject to air resistance proportional to the velocity is considered. In §4 the problem is of vertical motion away from the surface of the earth under varying gravity; air resistance is neglected and the maximum velocity attained is sufficiently small so that the distance moved from the surface is small in comparison with the earth's radius. These assumptions permit simplifying approximations to be used when calculating the distance traveled.

Fourth lecture. In §1 the general vectorial equation of motion of a particle which is both losing mass and gaining mass is obtained; it is of similar form and was given by Meščerskiĭ [Izvestiya St.-Peterburg. Politehn. Inst. 1, no. 1–2, 77–118 (1904)]. In §2 this is applied, as an approximation, to the motion of a reaction boat which expels the same amount of water as it takes in. The motion of a uniform chain falling over the edge of a table is studied as a further example on variable mass in §3.

Fifth lecture. In §1 the rectilinear motion of an aircraft with air-jet motor is considered; it is assumed that conditions are such that the particle theory can be applied. Thus, for example, the motion of the mass-centre in the aircraft is supposed negligible, and the motion of the air inside the aircraft is ignored. Resistance is taken to be proportional to velocity. The velocity and horizontal range are calculated. In §2 rectilinear motion of a particle under gravity is treated for a general law of air resistance, it being assumed that the rate of gaining mass is a constant multiple of the rate of losing mass. The optimum form of the function $f(t)$ is found in order to give the maximum range. In §3 the similar problem of finding the optimum $f(t)$ such that a

given distance is traversed in a minimum time is discussed. In both cases $f(t)$ is obtained, in the first place, as a function of the velocity.

Sixth lecture. This deals with the converse problem where from known external forces and motion the law of burning is to be determined. In §1 such a problem due to Meščerskiĭ (1897) is considered; it is assumed that s and v are known as functions of t, also $g=g(s)$ and the inclination θ of the trajectory (assumed constant). With these conditions $f(t)$ is obtained and various examples are considered. In §2 a similar problem is discussed where the motion is curvilinear and, because of the additional information, can be solved without integrations. In §3 the somewhat different type of problem of finding $f(t)$ and a force normal to the trajectory so that the particle shall describe a given curve is treated. The final section contains the 'semi-converse' problem of finding $f(t)$ and one of the two space coordinates when the other is given.

Seventh lecture. The first two sections deal with the general equations of motion. In §3 motion under a central force inversely proportional to the square of the distance is considered. Where a particle of constant mass would move in an ellipse, one of variable mass moves in a series of neighbouring spirals. In §4 an external central force $\mathbf{F}=-kMr^n\mathbf{r}/r$ is considered and in §5 there is described a method of changing variables to deal with the problem of two bodies of variable mass. In §6 the kinetic energy is given and in §7 the coefficient of useful action of a rocket is defined as the ratio of the work done by the jet to the kinetic energy, and is less than unity except when the velocities of motion and ejection are equal.

Eighth lecture. In this and the following lecture the results obtained for particles of variable mass are applied to bodies of variable mass, these bodies being regarded as systems of particles subject to external and internal forces. §1 is concerned with fundamental theorems and ideas; the velocities and accelerations of a system of particles relative to fixed and moving axes are obtained. In §2 the equations expressing the rate of change of linear momentum are obtained. §3 deals with the motion of the mass-centre, and §4 with the angular momentum of a body of variable mass. In §5 the equations of motion arising from the rate of change of angular momentum are obtained in general terms and special examples are discussed.

Ninth lecture. §1. Angular momentum equations relative to moving axes in the body [due to Agostinelli, Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **70**, 240–265 (1935)]. §2. Kinetic energy of a body of variable mass. In §3 the equations are obtained in generalised coordinates as a generalisation of Lagrange's equations and in §4 these are put into canonical form in terms of the variables p_i, q_j. When the absolute velocities of mass ejection vanish these equations are of the same form as for bodies of constant mass. It is stated that these last results were obtained in 1941 and published by the author in Učenye Zapiski Moskov. Gos. Univ. **122**, Mehanika **2** (1948) [unavailable].

R. A. Rankin (Birmingham).

***Lotkin, Mark. Ballistic functions related to the Newtonian law of resistance.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 31–35. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author explains the theory of the reduction to quadratures of the equations of the trajectory of a projectile under the Newtonian Law ("Square Law") of air resistance. He establishes and uses a quadrature formula, exact for polynomials of order up to five, and publishes several striking tables presenting numerical data for comparison. He shows how to handle related questions of determining parameters and initial conditions on the basis of observed performance of a projectile. The study and exposition are in connection with the handy, close approximations provided by extensive tables recently computed at Aberdeen, essentially similar to the brief Euler-Otto tables [Otto, Tafeln für den Bombenwurf, Berlin, 1842].

A. A. Bennett (Providence, R. I.).

Bade, W. L. Relativistic rocket theory. Amer. J. Phys. **21**, 310–312 (1953).

By application of relativistic mechanics to rockets Ackeret [Helvetica Phys. Acta **19**, 103–112 (1946); these Rev. **7**, 492] has obtained the differential equation of one-dimensional rocket motion and has found as a first integral a formula for calculating the final velocity of a rocket in an inertial frame $K(XYZT)$ in terms of the ratio of propellant weight to initial gross weight of the rocket and the exhaust velocity, this last being constant relative to the rocket. However, he has not discussed the problem of determining the integrated equation of motion $X=X(T)$ of the rocket in K, where T is the time read by a clock at rest in K. The author solves this last problem by determining X and T as functions of the proper time s of the rocket. It is assumed that K has been so chosen and the clocks in K and on the rocket have been so set that, when $s=0$, $X=T=dX/dT=0$.

E. Leimanis (Vancouver, B. C.).

Hydrodynamics, Aerodynamics, Acoustics

Truesdell, C. Two measures of vorticity. J. Rational Mech. Anal. **2**, 173–217 (1953).

La grande place que tiennent les mouvements irrotationnels dans la dynamique des fluides a conduit l'auteur à se poser le problème de caractériser, au moyen de grandeurs sans dimensions, l'importance relative du tourbillon dans un mouvement réel.

En se plaçant à un point de vue cinématique, l'auteur définit un premier nombre $\mathfrak{W}_K$, qui permet de comparer localement la vitesse de rotation à la vitesse moyenne de déformation. En désignant par w la grandeur du vecteur tourbillon $\mathbf{w}$, et par d_{ij} le tenseur des vitesses de déformation, ce nombre sans dimensions $\mathfrak{W}_K$ est défini par

$$\mathfrak{W}_K=\frac{w}{(2d^{ij}d_{ij})^{1/2}}$$

ce qui, après transformations, peut se mettre sous la forme

$$\mathfrak{W}_K=\left(\frac{\nabla^2 v^2-2\mathbf{v}\cdot\nabla^2\mathbf{v}}{w^2}-1\right)^{-\frac{1}{2}}$$

où v désigne la grandeur du vecteur vitesse $\mathbf{v}$.

Par ailleurs, la considération des équations de la dynamique des fluides conduit à évaluer l'importance qu'ont dans ces équations les termes qui dépendent du vecteur tourbillon. L'accélération d'une particule fluide étant donnée par

$$\rho\mathbf{a}=\rho\left[\frac{\partial\mathbf{v}}{\partial t}+\mathbf{w}\times\mathbf{v}+\operatorname{grad}\tfrac{1}{2}v^2\right]$$

il est introduit pour cela un deuxième nombre sans dimensions

$$\mathfrak{M}_D = \frac{|\mathbf{w}\times\mathbf{v}|}{|\partial\mathbf{v}/\partial E + \operatorname{grad} \frac{1}{2}v^2|}.$$

Ces définitions étant posées, le reste du travail est consacré à l'étude des valeurs et des propriétés de ces deux nombres, pour différentes classes de mouvements. Citons en particulier les cas suivants qui sont examinés: fluide incompressible; mouvements où le vecteur $\mathbf{w}\times\mathbf{v}$ est nul et pour lesquels la valeur de $\mathfrak{M}_K$ est liée au signe de $\nabla^2 v^2$; onde de Gerstner; mouvements de Poiseuille et mouvements à la Couette. Dans ce dernier cas, une condition nécessaire et suffisante de stabilité est que $\mathfrak{M}_D$ soit supérieur à un. Enfin, est étudié le mouvement rotationnel engendré par une onde de choc incurvée dans un écoulement uniforme et irrotationnel. C'est d'ailleurs là le probléme qui est à l'origine des recherches de l'auteur. *R. Gerber* (Toulon).

Van Mieghem, Jacques-M. Comments on the vorticity equation. Proc. Indian Acad. Sci. Sect. A. **37**, 204–212 (1953).

The author derives general differential equations for the rate of change of vorticity due to Beltrami [Opere matematiche, v. 2, Hoepli, Milan, 1904, pp. 202–379, see §7] and to the reviewer [Z. Angew. Math. Physik **2**, 109–114 (1951); these Rev. **12**, 761]. He discusses the form which these assume when simplifying assumptions appropriate to various meteorological situations are added.

C. Truesdell (Bloomington, Ind.).

Moreau, Jean-Jacques. Bilan dynamique d'un écoulement rotationnel. J. Math. Pures Appl. (9) **31**, 355–375 (1952); **32**, 1–78 (1953).

This memoir expands the author's three notes on the theory of vorticity [C. R. Acad. Sci. Paris **226**, 1420–1422 (1948); **228**, 1923–1925 (1949); **229**, 100–102 (1949); these Rev. **11**, 62, 871]. The discontinuity of vorticity across a vortex-sheet is examined and is shown to obey laws analogous to these for continuous distributions of vorticity. Most of the work concerns the behavior of the integrals

$$\mathbf{I} = \frac{1}{2}\int_V \mathbf{r}\times\mathbf{w}\,dv, \quad \mathbf{J} = -\frac{1}{2}\int_V r^2\mathbf{w}\,dv,$$

where $\mathbf{r}$ is the radius vector and $\mathbf{w}$ the vorticity vector. The author relates their time rates of change to the resultant force and moment of force acting on V, and discusses at length their application to a velocity field satisfying certain order conditions at infinity or in other ways especially restricted outside a finite region.

While the author claims $\mathbf{I}$ and $\mathbf{J}$ as new, they and their connection with linear and angular momentum are classical [e.g., Lamb, Hydrodynamics, 6th ed., Cambridge, 1932, see §152]. There is a total absence of references to the extensive non-French literature on rotational motions.

C. Truesdell (Bloomington, Ind.).

Darwin, Charles. Note on hydrodynamics. Proc. Cambridge Philos. Soc. **49**, 342–354 (1953).

The author studies the actual trajectories of fluid particles in certain motions of an incompressible inviscid fluid. When a solid body moves through the fluid, it induces a "drift" such that the final positions of the particles are further on than those from which they started. The "drift-volume" enclosed between the initial and final positions is equal to the volume corresponding to hydrodynamic mass, that is, the mass of fluid to be added to that of the solid in calculating its kinetic energy. This result is proved quite generally, and thus it is shown for the first time, as far as the reviewer knows, that the concept of hydrodynamic mass is not a mathematical fiction but a genuine physical phenomenon in which an amount of fluid corresponding to the hydrodynamic mass is being really carried along by the solid body. The author further considers the trajectories of the particles surrounding (or inside) a rotating elliptic cylinder and shows that the fluid particles slowly drift around the body, even though the motion is irrotational and without circulation. The present work shows that the ideal fluid has features very like those of a real fluid. The passage of a solid body is accompanied by a "wake" and in the case of a rotating body a certain amount of fluid may be entrained permanently in an "eddy".

L. M. Milne-Thomson (Greenwich).

Morgan, G. W. Remarks on the problem of slow motions in a rotating fluid. Proc. Cambridge Philos. Soc. **49**, 362–364 (1953).

Stewartson, in his solution of the problem of a sphere in slow motion along the axis of a rotating inviscid incompressible fluid [same Proc. **48**, 168–177 (1952); these Rev. **13**, 997], assumes initial conditions which state that the motion of the fluid is initially that of a uniform stream which is nowhere disturbed by the sphere except on the latter's surface. The author of this note considers these conditions as inappropriate to the problem, recommends instead that the initial motion relative to the rotating system be the irrotational motion with the given boundary conditions, and shows that Stewartson's solution actually does satisfy the latter condition rather than the one apparently imposed.

D. Gilbarg (Bloomington, Ind.).

Stewartson, K. A weak spherical source in a rotating fluid. Quart. J. Mech. Appl. Math. **6**, 45–49 (1953).

Using the same method as in his solution of the problem of slow motion of a sphere in a rotating inviscid incompressible fluid [Proc. Cambridge Philos. Soc. **48**, 168–177 (1952); these Rev. **13**, 997], the author here determines the ultimate flow when a weak spherical source is placed in the fluid. He obtains the interesting result, to be contrasted with the spherical symmetry when the body of fluid is not rotating, that inside the cylinder C circumscribing the sphere, and having its axis on the axis of rotation, the perturbation in the velocity of the fluid due to the source is ultimately wholly parallel to the axis of rotation and singular on C, while outside C the perturbation consists of a transverse velocity about the axis which is also singular on C. In addition to this steady behavior, the velocity components are oscillatory on the sphere and on its axis, while on C they grow without bound. This problem was also treated by Grace [Proc. Roy. Soc. London. Ser. A. **105**, 532–543 (1924)] who obtained some of the author's subsidiary results by a different method. *D. Gilbarg*.

Birkhoff, Garrett. Induced mass with variable density. Quart. Appl. Math. **11**, 109–110 (1953).

The concept of induced mass, and some of its properties, are extended to the case of an incompressible fluid of variable density. The proofs parallel closely those given recently by the author [same Quart. **10**, 81–86 (1952); these Rev. **13**, 877] for the case of free boundaries. (Author's summary.)

D. Gilbarg (Bloomington, Ind.).

Schmieden, C. Der Aufschlag von Rotationskörpern auf eine Wasseroberfläche. Z. Angew. Math. Mech. **33**, 147–151 (1953). (English, French, and Russian summaries)

Discussion of the impact of a body on a free surface at water entry, based on approximation of the flow at each instant by the steady flow past a circular plate.

P. R. Garabedian (Stanford, Calif.).

***Fuchs, Robert A. On the theory of short-crested oscillatory waves.** Gravity Waves, pp. 187–200. National Bureau of Standards Circular 521, U. S. Government Printing Office, Washington, D. C., 1952. $1.75.

A second-order approximation for the propagation of doubly periodic waves is carried out and compared with the corresponding theory (Stokes) of singly periodic waves. The propagation of an initial elevation of the surface is treated by the method of stationary phase and the method applied to an initial doubly periodic elevation over a square region. *J. V. Wehausen* (Providence, R. I.).

***Biesel, F. Study of wave propagation in water of gradually varying depth.** Gravity Waves, pp. 243–253. National Bureau of Standards Circular 521, U. S. Government Printing Office, Washington, D. C., 1952. $1.75.

The author treats (two-dimensionally) gravity waves of a perfect fluid over a bottom of variable depth $h(x)$. He finds a function $\phi(x, y, t)$ satisfying the linearized free surface and bottom conditions and also $\Delta\phi=0$ provided terms of higher than the first degree in $\alpha=h'(x)$ are discarded. Using this function ϕ as a velocity potential and changing to Lagrangian coordinates, he finds expressions for the free surface and the particle orbits. The form of the wave profile is discussed and results of calculations for a wave progressing and breaking on a flat beach of 1 to 10 slope are given. Results are also given for a similar calculation when terms in the square of the wave amplitude are retained.

J. V. Wehausen (Providence, R. I.).

Dressler, R. F., and Pohle, F. V. Resistance effects on hydraulic instability. Comm. Pure Appl. Math. **6**, 93–96 (1953).

C'est une étude de la stabilité d'un écoulement uniforme avec surface libre dans un canal rectangulaire, incliné. Il est supposé exister une résistance de paroi qui se traduit par une force de masse, fonction de la vitesse et de la profondeur. La stabilité est étudiée par le calcul des perturbations en généralisant la méthode qui a été utilisée par Jeffreys pour le cas où la résistance obéit à la formule de Chézy. L'auteur montre la possibilité de perturbations périodiques et il aboutit à des critères de stabilité qui coïncident avec ceux obtenus par des voies différentes par Vedernikov et par Craya. Les résultats mettent en évidence le fait que l'instabilité nécessite que la force retardatrice dépende à la fois de la vitesse et de la profondeur. *R. Gerber* (Toulon).

Polyahov, N. N. On the pressure distribution on the surface of a profile moving unsteadily. Doklady Akad. Nauk SSSR (N.S.) **87**, 901–904 (1952). (Russian)

The velocity potential function for a profile in an unsteady plane incompressible flow is of the form

$$\Phi(t, x_0, y_0)=u_s(t)\Phi_1(x, y)+v_s(t)\Phi_2(x, y) +\omega(t)\Phi_3(x, y)+\Gamma(t)\Phi_\Gamma(x, y)+\Phi_s(t, x, y).$$

Here x_0, $y_0(x, y)$ are cartesian coordinates for axes fixed in space (rigidly attached to the profile); u_s, v_s is the velocity of $x=y=0$ relative to the xy-system; ω is the angular velocity of and Γ the circulation about the profile; and Φ_s is the potential of the cast-off vorticity or wake. Lagrange's formula $(p-p_\infty)/\rho+\frac{1}{2}(u^2+v^2)+\partial\Phi/\partial t=0$ yields the pressure. Only the determination of Φ_s presents a problem, approximately solved as follows by the author. Approximate the actual wake by that for a motion at zero lift. Suppose $x+iy=\zeta+R^2/\zeta+\sum_1^\infty(a_n+ib_n)\zeta^{-n}$ maps the profile onto $|\zeta|=|\xi+i\eta|=R$, and that the zero lift wake's image is $\eta=0$, $\xi>1$. On the wake the author sets $y_s=\sum_1^\infty b_n\xi^{-n}$ and approximates $x_s=\xi+R^2/\xi$. The Kutta-Joukowski condition is applied to determine $\Gamma=\Gamma_b+\Gamma'$ in terms of $\partial\Phi_1/\partial\theta$, etc., at the trailing edge, where Γ_b is the circulation that would occur in the absence of the wake. Then

$$\Gamma'=2R\int_1^\infty \gamma_1(\bar{x}_s, t)[(\bar{x}_s+1)^{1/2}(\bar{x}_s-1)^{-1/2}-1]d\bar{x}_s$$

where $\bar{x}_s=x_s/2R$ and $2R\gamma_1=-d\Gamma/d\bar{x}_s$. Also, at $\zeta=Re^{i\theta}$ on the profile $\partial\Phi_s/\partial t=\int_0^\theta R\partial V_0/\partial t d\theta$, where

$$\pi V_0(\theta, t)=\Gamma'/2R+\int_1^\infty \gamma_1(\bar{x}_s, t)[1-(\bar{x}_s^2-1)^{1/2}/(\bar{x}_s+\cos\theta)]d\bar{x}_s.$$

The author sketches an application to a harmonically oscillating profile for which $\Gamma(x_s, t)$ is known, and also indicates a method for taking into account the neglected terms in x_s. *J. H. Giese* (Havre de Grace, Md.).

Fettis, Henry E. Regarding the computation of unsteady air forces by means of Mathieu functions. J. Aeronaut. Sci. **20**, 437–438 (1953).

Falkner, V. M. The solution of lifting-plane problems by vortex-lattice theory. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2591** **(10,895)**, 30 pp. (1953).

Here the bound-vortex distribution over the wing is expressed in a double series and the corresponding downwash is calculated. A finite number of terms is employed and the downwash is matched to the given wing slope at a corresponding number of stations on the planform area. For example, 21, 84, or 126 such "control points" may be used. The appropriate matrices for these approximations are shown. The method is essentially the same as proposed by the same author in an earlier report [same Rep. and Memoranda no. **1910** (1943)]. This report summarizes the method and the results of several years' experience with it. A set of solutions for an equilateral delta-wing is included to illustrate a number of points. *W. R. Sears* (Ithaca, N. Y.).

van de Vooren, A. I. The generalization of Prandtl's equation for yawed and swept wings. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 121**, i+22+ii pp. (1952).

The downwash distribution is first expressed by an integral over the bound and trailing vortex sheets in the usual way. The vortex strengths are then expressed in Taylor series in the spanwise variable, whereupon the downwash integral appears as a sum of integrals. The orders of magnitude of these terms for a wing of large aspect ratio ϵ^{-1} are studied and only those of order ϵ are retained; i.e., $O(\epsilon^2 \ln \epsilon)$ is neglected. Straight yawed wings and sweptback wings are then considered in detail. The result in each case is an integral equation whose solution in practical cases is postponed to later reports. The downwash at any spanwise station is expressed as the sum of the two-dimensional distribution, an induced camber, and the induced incidence. The results

are not expected to be useful for the regions near wing tips or near the root sections of sweptback wings, because of the approximations made. *W. R. Sears* (Ithaca, N. Y.).

★Reissner, Eric. A problem of the theory of oscillating airfoils. Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 923–925. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author's approximate theory [NACA Tech. Note no. **1194** (1947); these Rev. **8**, 542] for oscillating wings of finite aspect ratio in incompressible flow is applied to the determination of the circulation along an oscillating wing of infinite span that exhibits a periodic deformation in the spanwise coordinate. The result is expressed in terms of known functions and therefore is somewhat simpler to use than the earlier solution of the same problem given by Sears [Proc. 5th Internat. Congress. Appl. Mech., Cambridge, Mass., 1938, Wiley, New York, 1939, pp. 483–487], in which numerical quadrature was necessary.

J. W. Miles (Los Angeles, Calif.).

Jaumotte, André. Généralisation de la formule de Kutta et Joukowski aux grilles d'ailes en fluide réel incompressible. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 1158–1162 (1952).

The author extends the familiar formula of Kutta and Joukowski, which relates the lift to the circulation of an isolated airfoil in an ideal fluid, to the airfoils making up a cascade in an incompressible viscous fluid.

The reviewer remarks that it is necessary to define the circulation by a contour going to infinite distance behind the cascade. Thus, in a viscous fluid, the relation between this circulation and the circulation measured at or near the airfoil is not clear. *W. R. Sears* (Ithaca, N. Y.).

Betz, A. Zur Berechnung von Gitterströmungen bei einigermassen grossem Schaufelabstand. Z. Angew. Math. Mech. **33**, 113–116 (1953). (English, French, and Russian summaries)

Approximate formulas are obtained by series expansion for the flow field due to an infinite cascade of vortices with one vortex missing. These can be generalized to apply to a corresponding cascade of airfoils. They are useful for the correction of single-airfoil characteristics to the case of an infinite cascade of airfoils. *W. R. Sears.*

Litwiniszyn, J. Generalization of some equations of hydrodynamics. Prace Mat.-Fiz. **48**, 1–26 (1952).

The author takes the usual form for the stress tensor $t^i{}_j$ in a viscous fluid, viz.,

$$t^i{}_j = -p\delta^i{}_j + \lambda d^k{}_k \delta^i{}_j + 2\mu d^i{}_j;$$

d_{ij} is the symmetric part of $v_{i,j}$, v_i being the velocity vector. This formula in the special case when $\mu=$const. and $3\lambda+2\mu=0$ he takes as the definition of a viscous fluid in a Riemannian space of n dimensions, and apparently he assumes $n=3$. He works out the form which several of the classical equations of hydrodynamics assume in virtue of the fact that covariant derivatives are not in general commutable. These involve $g^{ij}v^k R_{jk}$, where g^{ij} is the metric tensor and R_{jk} is the contracted curvature tensor. [Most of the results are purely formal and seem evident to the reviewer. As is plain, those which are derived in Euclidean space without interchanging the order of derivatives are valid also in any Riemannian space, e.g., the Bjerknes formula.] The author discusses also surface tension and viscous dissipation. There is an appendix containing standard tensor manipulations. *C. Truesdell* (Bloomington, Ind.).

Ballabh, Ram. On two-dimensional superposable flows. J. Indian Math. Soc. (N.S.) **16** (1952), 191–197 (1953).

The author continues his investigations of superposable flows [Proc. Benares Math. Soc. (N.S.) **2**, 69–79 (1940); **4**, 27–31 (1943); J. Indian Math. Soc. (N.S.) **7**, 36–41 (1943); these Rev. **3**, 283; **5**, 247; **5**, 79], two flows in the same external force field being superposable if the resultant of the sum of their two velocity fields is itself the velocity field of a flow in that force field. If ψ_1, ψ_2 are the stream functions of two plane flows of an incompressible viscous fluid, and ζ_1, ζ_2 the respective vorticities, the two flows are superposable if and only if

$$(*)\qquad \partial(\psi_1,\zeta_2)/\partial(x,y)+\partial(\psi_2,\zeta_1)/\partial(x,y)=0.$$

The author gives a partial characterization of the special class of superposable flows in which one of the two flows is steady and irrotational, say $\zeta_2=\Delta\psi_2=0$; in this case $\zeta_1=f(\psi_2)$ as is evident from (*), and the vorticity is therefore constant on the streamlines of a potential flow. Flows with this property have been fully described by G. Jeffery [Philos. Mag. (6) **29**, 455–465 (1915)]. The author makes several remarks about self-superposable flows in which, as (*) shows, vorticity is constant on streamlines. These flows have been studied extensively also by others [cf. R. Berker, Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible, Taffin-Lefort, Paris-Lille, 1936]. *D. Gilbarg* (Bloomington, Ind.).

Prem Prakash. General steady flow superposable on a constant velocity. Ganita **3**, 91–93 (1952).

This paper determines the plane flows of a viscous fluid that are superposable in the sense of Ballabh [see the preceding review] on a uniform flow. The results are a special case of those in the paper reviewed above. *D. Gilbarg.*

Ghosh, N. L. A note on the transition from viscous to perfect fluid flow. Proc. Nat. Inst. Sci. India **18**, 467–472 (1952).

By a purely local argument, without considering boundary conditions, the author claims to prove that a plane viscous flow with nonconstant vorticity cannot reduce to a potential flow in the limit as viscosity tends to zero. The proof is in error, and simple counterexamples show that the result is untrue. *D. Gilbarg* (Bloomington, Ind.).

Ghosh, N. L. Note on a class of exact solutions of the two-dimensional flow problem for a viscous incompressible fluid. Proc. Nat. Inst. Sci. India **18**, 473–479 (1952).

This paper treats the two semi-inverse problems of determining the plane flows of a viscous incompressible fluid for which (1) the vorticity is constant on the streamlines of a potential flow, (2) the streamlines coincide with the streamlines of a potential flow. The author gives a partial solution to these problems, both of which have been completely solved, the first by G. Jeffery [Philos. Mag. (6) **29**, 455–465 (1915)] who showed that the curves of constant vorticity are either parallel straight lines or concentric

circles, and the second by G. Hamel [Jber. Deutsch. Math. Verein. **25**, 34–60 (1916)] who showed that the only solutions are his well-known spiral flows. *D. Gilbarg.*

Stewartson, K. On the flow between two rotating coaxial disks. Proc. Cambridge Philos. Soc. **49**, 333–341 (1953).

The author investigates, both experimentally and theoretically, the nature of the steady rotationally symmetric flow of a viscous incompressible fluid between two parallel infinite disks rotating with uniform angular velocity about a common axis. In a qualitative discussion of the same problem, Batchelor [Quart. J. Mech. Appl. Math. **4**, 29–41 (1951); these Rev. **13**, 82] concluded that, at large Reynolds numbers, when the two disks rotate in the same sense the main body of the fluid also rotates in that sense with an angular velocity between those of the two disks, but that when the disks rotate in opposite senses the main body of the fluid is in two parts with different angular velocities. The experimental results reported by the author of the present paper are in agreement with the first conclusion, but at variance with the second. In the latter case, and also in the case that one of the disks is at rest, he finds instead that the main body of fluid is at rest. The author, using the same equations as Batchelor, presents a well supported, although not rigorously established, boundary layer type theory which accounts for all the observed results. *D. Gilbarg.*

Berker, Ratip. Sur les forces exercées par un fluide visqueux sur un obstacle. Rend. Circ. Mat. Palermo (2) **1**, 260–280 (1952).

Let a flow that is uniform at infinity be termed regular of order k if the velocity components and their first and second derivatives approximate their respective limit values within $o(r^{-k})$, $o(r^{-k-1})$, $o(r^{-k-2})$ as $r\to\infty$. In this paper the author shows that a three-dimensional flow of a viscous fluid past a finite body with adherence on the boundary can be regular of order k only if $k<2$. He shows also that the d'Alembert paradox, which was proved for viscous fluids by earlier investigators under the assumption of sufficient regularity at infinity, holds for $k\geqq 2$; since flows with this order of regularity are the only ones for which it has been proved, the d'Alembert paradox in viscous fluids applies thus far only to non-existent flows, as we would expect. The author establishes analogous results for the slow flows of Stokes, the Oseen modification of the Stokes flows, and also for plane flows and flows at rest at infinity. A similar but somewhat more special result has been obtained by Cabannes [Bull. Soc. Math. France **80**, 37–46 (1952); these Rev. **14**, 595] for flows expansible in negative powers of r at infinity. *D. Gilbarg* (Bloomington, Ind.).

Wakiya, Shôichi. A spherical obstacle in the flow of a viscous fluid through a tube. J. Phys. Soc. Japan **8**, 254–257 (1953).

The author considers the effect of a sphere, located at the center of the pipe, in the Poiseuille flow through a circular pipe. By the Stokes approximation, a solution is found, which satisfies the surface conditions and reduces to the Poiseuille flow, both far up- and down-stream of the sphere. It is concluded that, for the same flux through the pipe, the presence of the sphere requires additional pressure difference which increases rapidly with the ratio of the radii of the sphere and the pipe. The drag on the sphere is also increased. *Y. H. Kuo* (Pasadena, Calif.).

✶Rouse, Hunter, and Hsu, Hsieh-ching. On the growth and decay of a vortex filament. Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 741–746. The American Society of Mechanical Engineers, New York, N. Y., 1952.

This paper concerns the problem of growth and decay of a vortex whose motion is assumed to be circular and depends on time and radial distance from the center of rotation. If the motion was initially at rest, the solution gives infinite kinetic energy at all time. To remove these difficulties, the authors divide the motion into generating and dissipating stages such that the subsequent dissipating motion begins with a definite condition. With this modification, the solution yields finite results which in some instances can be compared with observation. It appears that the authors are unaware of the work of previous authors such as Terazawa, Levy, and others. [See Bateman, Bull. Nat. Res. Council no. **84**, 153–332 (1932), pp. 212–221]. *Y. H. Kuo.*

Hancock, G. J. The self-propulsion of microscopic organisms through liquids. Proc. Roy. Soc. London. Ser. A. **217**, 96–121 (1953).

The author presents a new approach to the problem of determining the motion of filamentary organisms of small diameter which propel themselves by sending disturbances of prescribed shape down their length. The theoretical study of self-propulsion of micro-organisms was initiated by G. I. Taylor [same Proc. **209**, 447–461 (1951); **211**, 225–239 (1952); these Rev. **13**, 596; **14**, 104] and extended by M. Lighthill [Comm. Pure Appl. Math. **5**, 109–118 (1952); these Rev. **14**, 424]. Because of the small Reynolds numbers involved, the mathematical details in these investigations and in the paper under review are based on neglect of the inertial terms in the Navier-Stokes equations. The author's approach consists in satisfying the two conditions of adherence on the boundary and of finite energy in the velocity field by placing appropriate distributions of singularities along the filament axis. The singularities used, which are suggested by the Stokes theory of slow motion of a sphere, are an ordinary doublet and a second singularity peculiar to the equations of slow motion. The author carries out calculations based on his general method for several special cases, among them the case of an infinite filament down which lateral plane waves propagate. In the common range of validity of the theories due to Taylor and the author for this case the two give the same result that the velocity of propulsion is approximately $2\pi^2U\eta^2/\lambda^2$, where η is the amplitude of the wave, λ the wavelength, and U the velocity of wave propagation. The author also calculates motion due to spiral waves and to longitudinal waves propagating along a filament. *D. Gilbarg* (Bloomington, Ind.).

✶Poritsky, H. The collapse or growth of a spherical bubble or cavity in a viscous fluid. Proceedings of the First U. S. National Cóngress of Applied Mechanics, Chicago, 1951, pp. 813–821. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The equations of spherically symmetric viscous flow are derived and applied to the expansion or collapse of a bubble in a viscous fluid. A nonlinear equation of the form

$$(*)\qquad \beta d^2\beta/d\tau^2+(3/2)(d\beta/d\tau)^2+(C/\beta)(d\beta/d\tau)+1=0$$

is obtained to describe the motion of the bubble wall. Devices for the numerical integration of this equation are discussed in detail. Further numerical results are given when

the effect of surface tension is considered. It is concluded that the time of collapse of a bubble may be finite or infinite when surface tension is ignored, but is always finite when surface tension is considered. *E. Pinney.*

⁎Shu, S. S. Note on the collapse of a spherical cavity in a viscous incompressible fluid. Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 823–825. The American Society of Mechanical Engineers, New York, N. Y., 1952.

A Poincaré-Bendixon topological analysis of Poritsky's equation [(*) in the preceding review] is made. It is concluded that when surface tension is ignored, the time of collapse of a bubble is finite or infinite according as C (which is proportional to the coefficient of viscosity) is less than or greater than $\sqrt{6}$. *E. Pinney* (Berkeley, Calif.).

Sowerby, L., and Cooke, J. C. The flow of fluid along corners and edges. Quart. J. Mech. Appl. Math. **6**, 50–70 (1953).

This paper extends Rayleigh's idea to study the flow along corners and edges. In Part I, Rayleigh's approximate method for the case of the flow along the outside of a circular cylinder is tested by solving both the approximate and the boundary-layer problems. As expected, the main features of the solution given by both methods do agree. It is interesting to note that the skin-friction coefficient is expanded as a Taylor series in $Re^{-1/2}$, Re being the Reynolds number.

In Part II, Rayleigh's problem for a wedge is solved by the analogy of heat conduction. For the case of wedge angle $3\pi/2$ constant velocity lines are calculated. It is concluded that (1) there are two "boundary-layer thicknesses", viz., the "main" thickness and the width of the region around the corner, and they are both of the same order of magnitude; (2) the width of the region increases as the wedge angle decreases from 2π; (3) the whole boundary layer decreases in size as the corner is approached if the wedge angle is greater than π, and increases if the angle is less than π; (4) the skin-friction decreases steadily from the flat-plate value to zero at the corner for angles less than π, and for angles greater than π, it increases steadily from the flat-plate value. *Y. H. Kuo* (Pasadena, Calif.).

Probstein, Ronald F. On a solution of the energy equation for a rotating plate started impulsively from rest. Quart. Appl. Math. **11**, 240–244 (1953).

The laminar thermal boundary layer on a rotating disc for an incompressible viscous fluid is investigated assuming initially impulsive motion. Solutions are obtained for small values only on time and radius. The work represents an extension of the results of Nigam [same Quart. **9**, 89–91 (1951); these Rev. **12**, 764] on velocity distribution for the same configuration. The energy equation is reduced to a pair of simultaneous non-linear ordinary differential equations which are solved using the same functions arising in Nigam's analysis. Results are given for the cases of zero wall temperature gradient and of parabolic radial wall temperature variation. *N. A. Hall* (Minneapolis, Minn.).

Stewartson, K. On the flow downstream of separation in an incompressible fluid. Proc. Cambridge Philos. Soc. **49**, 561–569 (1953).

The author continues the boundary layer theory beyond the point of separation by assuming that the solution downstream has a free stream line. Two solutions are obtained: one having a continuous pressure gradient, the other having zero pressure gradient. Thus, the boundary layer theory is made to predict a state of affairs near separation which is at least in qualitative agreement with experiment. *C. C. Lin* (Cambridge, Mass.).

de Possel, René, et Valensi, Jacques. Sur le sillage d'une plaque perméable. C. R. Acad. Sci. Paris **236**, 2211–2213 (1953).

The authors consider the flow through and about a finite vertical porous plate in a horizontal uniform stream. The flow is said to be rotational in the wake and irrotational elsewhere. The rotational part of the velocity is assumed to vary in the transverse direction only. The only example worked out, however, seems only vaguely related to the original problem, since the velocity field is simply a superposition of two singularities at the tips of the plate and a uniform stream. The flow then is irrotational with two lines of discontinuity trailing behind the plate. *Y. H. Kuo.*

⁎Brown, W. Byron. Exact solutions of the laminar boundary layer equations for a porous plate with variable fluid properties and a pressure gradient in the main stream. Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 843–852. The American Society of Mechanical Engineers, New York, N. Y., 1952.

This paper deals with the problems of boundary-layer flow over a porous plate under the assumptions that the thermal properties of a gas such as viscosity, thermal conductivity, heat capacity, etc., vary with the absolute temperature according to power laws. In the cases considered by Falkner and Skan, in which the exterior velocity varies with powers of the distance along the plate, solutions are carried out under various conditions of suction rate and temperature ratio. *Y. H. Kuo* (Pasadena, Calif.).

⁎Tifford, Arthur N. An exact thermal solution in laminar viscous flow. Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 783–788. The American Society of Mechanical Engineers, New York, N. Y., 1952.

It was known that in the case of a main stream velocity which is proportional to the distance from the stagnation point along the surface, the boundary layer solution is also the exact solution. For this velocity field, the thermal problem can be exactly solved with arbitrary surface temperature distribution. With a view to bringing out the nature of the boundary-layer approximations, the exact and approximate solutions are compared along the stagnation line (temperature varies in the third direction as well). This leads the author to conclude that if the fourth or any higher even derivative is large along this line, boundary-layer theory gives erroneous results. Further, it is shown that the heat-transfer coefficient includes an extra term which is inversely proportional to the Reynolds number. *Y. H. Kuo* (Pasadena, Calif.).

Lessen, Martin. A remark on the stability of the laminar boundary layer in a compressible fluid. J. Aeronaut. Sci. **20**, 500 (1953).

Burgers, J. M. Some considerations on turbulent flow with shear. I, II. Nederl. Akad. Wetensch. Proc. Ser. B. **56**, 125–136, 137–147 (1953).

The author sets up a linear model for studying turbulent fluctuations in shear flow, with the idea that this gives an

adequate description of those features of the turbulence which depends on the coupling between turbulent motion and mean flow. With the help of further adequate assumptions, a formula is obtained for the Reynolds shear. For a suitable range, this leads to the same results as obtained in Kármán's similarity theory. The constants of proportionality in the theory are evaluated, and are found to be of the right order of magnitude. Development of a more refined theory is suggested. *C. C. Lin* (Cambridge, Mass.).

Nardini, Renato. Due teoremi di unicità nella teoria delle onde magneto-idrodinamiche. Rend. Sem. Mat. Univ. Padova 21, 303–315 (1952).

Ce travail est consacré aux ondes magnéto-hydrodynamiques dans un fluide homogène, incompressible, visqueux, électriquement conducteur et soumis à un champ magnétique. L'auteur démontre l'unicité de la solution des équations qui régissent le problème convenablement précisé par des conditions initiales et des conditions aux limites. On suppose que la solution satisfait à certaines conditions de régularité. L'unicité est démontrée d'abord en négligeant le courant de déplacement, puis en tenant compte de ce courant. Le domaine occupé par le fluide est supposé borné, mais les résultats obtenus sont étendus au cas où ce domaine s'étend à l'infini. L'auteur se sert de la méthode introduite par D. Graffi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 12, 129–135 (1930)]. *R. Berker.*

Nardini, Renato. Due teoremi di unicità nella magnetodinamica dei fluidi compressibili. Boll. Un. Mat. Ital. (3) 7, 403–411 (1952).

Ce travail traite du même problème que le travail analysé ci-dessus, la seule différence étant que le fluide est maintenant supposé non plus incompressible, mais compressible et barotrope. L'unicité de la solution est démontrée par la même méthode; certains résultats obtenus par D. Graffi [J. Rational Mech. Anal. 2, 99–106 (1953); ces Rev. 14, 598] sont utilisés. Les conditions initiales, les conditions aux limites, et les conditions de régularité sont identiques (complétées par des conditions analogues pour les grandeurs non-mécaniques) à celles du mémoire de Graffi qui vient d'être cité. *R. Berker* (Istanbul).

Ludford, G., Polachek, H., and Seeger, R. J. On unsteady flow of compressible viscous fluids. J. Appl. Phys. 24, 490–495 (1953).

The authors present a method for numerical treatment of the one-dimensional unsteady motion of a viscous heat-conducting gas that is based on replacing the continuum equations by equations of motion for a finite system of particles, in which neighboring particles are subject to interactions approximating the stresses in the original gas. (A similar approach to this problem was taken by von Neumann in a wartime report.) The paper discusses three sample calculations, details of which were carried out on a digital computer, for two initial and boundary value problems describing gas motion in a tube. Two of the calculations involved 99 moving particles and the other 39. The authors find in their first calculation that for fluids of low viscosity their finite difference procedure wipes out the effect of viscosity, and also introduces spurious effects, when the basic space interval of integration is large compared to the shock thickness. They overcome these difficulties by executing their remaining calculations with unrealistically high values of viscosity, with the effect that the shock thicknesses are then sufficiently large for proper behavior of the calculations. The paper concludes with a proof that the finite-difference procedure is stable provided the time and space intervals of integration satisfy a certain inequality.
D. Gilbarg (Bloomington, Ind.).

*v. **Mises, R. On some topics in the fundamentals of fluid flow theory.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 667–671. The American Society of Mechanical Engineers, New York, N. Y., 1952.

English translation of Österreich. Ing.-Arch. 6, 77–85 (1952); these Rev. 14, 328. *D. Gilbarg.*

Pack, D. C. A note on the unsteady motion of a compressible fluid. Proc. Cambridge Philos. Soc. 49, 493–497 (1953).

A solution is found in closed form for the one-dimensional motion of an ideal, inviscid, and isentropic gas with ratio of specific heats $(2m+3)/(2m+1)$, m being any positive integer, when the gas is supposed to start from rest with an inhomogeneous temperature distribution. An example is given which is of interest in the study of interstellar gas clouds. A cloud is assumed to be initially at rest in contact with a vacuum; the Riemann invariant r is assumed to increase linearly for a finite distance into the cloud (from zero at the surface), beyond which it remains constant. The r-characteristics from the inhomogeneous strip meet simultaneously in the distance-time plane at a point of the gas surface; the surface does not move until this happens, when it acquires instantaneously a finite non-zero velocity. A numerical estimate is made of the time during which the surface remains at rest for an actual gas cloud under the above conditions. In the subsequent development the gas motion takes the form of a growing simple wave. (Author's abstract.) *Y. H. Kuo* (Pasadena, Calif.).

Crocco, Luigi. Transformations of the hodograph flow equation and the introduction of two generalized potential functions. NACA Tech. Note no. 2432, 81 pp. (1951).

The hodograph equations of motion in inviscid compressible fluid are given in a symmetrical form in terms of magnitudes of velocity and mass velocity. Some new relations are shown along the lines of approach of Bergman and of Bers and Gelbart. So far no simple useful solutions seem within reach of the method. Some improved approximations in the subsonic and supersonic regimes are indicated in the discussion of Chaplygin-Kármán-Tsien approximations. They do not seem practical for calculation in terms of Mach number as given in the figures, although the ideas have been developed independently by other investigators. Some transonic approximation is suggested.

Two generalized potential functions are introduced in terms of the complex velocity and mass velocity. Some advantages over the conventional approach are claimed. An example is shown with the approximation of Chaplygin-Kármán-Tsien. *C. C. Chang* (College Park, Md.).

Chang, Chieh-Chien, and O'Brien, Vivian. Some exact solutions of two-dimensional flows of compressible fluid with hodograph method. NACA Tech. Note no. 2885, 63 pp. (1953).

La fonction de courant d'un mouvement plan potentiel de fluide incompressible étant exprimée par une série suivant les puissances de la grandeur q de la vitesse, Chaplygin a montré, comme il est bien connu, que de ce mouvement, on peut déduire un mouvement plan potentiel de fluide com-

pressible en remplaçant les puissances de q par certaines fonctions hypergéométriques (fonctions de Chaplygin). Les auteurs du présent travail proposent de classer les mouvements qui peuvent être obtenus par cette méthode selon la position et le nombre des singularités dans le plan de l'hodographe. Ils obtiennent par la méthode en question les écoulements autour d'un obstacle en forme de coin convexe; dans ces écoulements il y a dans le plan de l'hodographe une seule singularité située à l'origine. Pour quatre cas correspondant à quatre valeurs différentes de l'angle du coin, les auteurs ont effectué les calculs numériques; les écoulements sont analysés en détail et plusieurs particularités nouvelles sont mises en lumière. Les auteurs donnent ensuite, en utilisant toujours la méthode de Chaplygin, l'écoulement dans un canal symétrique dont la section se contracte; il y a deux singularités dans le plan de l'hodographe, à savoir une source positive et une source négative. Dans un cas concret les résultats sont encore conduits jusqu'aux calculs numériques. Les auteurs ont amplifié les tables publiées par V. Huckel [NACA Tech. Note no. **1716** (1948); ces Rev. **10**, 329] des fonctions de Chaplygin et reproduisent les tables qu'ils ont obtenues. *R. Berker* (Istanbul).

Cabannes, Henri. Contribution à l'étude théorique des fluides compressibles. Ecoulements transsoniques. Ondes de choc. Ann. Sci. Ecole Norm. Sup. (3) **69**, 1–63 (1952).

Since the completion of this paper in 1949 its results have been extended substantially by the author and others; prior to publication it was summarized in four notes [C. R. Acad. Sci. Paris **229**, 102–104, 492–493, 510–511, 923–925 (1949); these Rev. **11**, 221, 279, 271, 479]. The author first presents a method for generating plane irrotational flows. His solutions, in hodograph variables, depend on an arbitrary function and a double infinity of constants, and include the Chaplygin solutions as special cases. The author applies his method to a Tricomi gas to derive exact solutions describing transsonic channel-like flows [cf. also Tomotika and Tamada, Quart. Appl. Math. **7**, 381–397 (1950); these Rev. **11**, 275]. The remainder of the paper is concerned with shock waves. The author begins with a general characterization of stationary flows in which the entropy jump across a shock is constant. He constructs several examples of such flows, among which is the flow formed by joining any two members of the same family of plane irrotational spiral flows along a common curve of constant speed. The author treats the problem of detached shock waves in a uniform flow by the method of power series expansions in the neighborhood of the shock vertex. He presents here the quadratic approximation to the solution of the plane and three-dimensional problem, and has since extended this result by calculating axially symmetric flows in which the approximation is correct to fourth order [Recherche Aéronautique no. **21**, 3–7 (1951); C. R. Acad. Sci. Paris **232**, 686–687 (1951); these Rev. **13**, 180; **12**, 553]. In the problem of the attached plane shock wave the author derives by his own method using intrinsic coordinates the relations connecting the curvature of the shock wave and its derivatives with those of the streamline [cf. also Thomas, J. Math. Physics **27**, 279–297 (1949); **28**, 62–90, 153–172 (1949); these Rev. **10**, 494, 758; **11**, 479]. The theory of the interesting singular case when the flow behind the shock is subsonic and the shock curve does not admit an analytic representation at the vertex is outlined in a note added in proof and is presented more fully in later work of the author [Publ. Sci. Tech. Ministère de l'Air, Paris, no. **250**, 181–196 (1951); these Rev. **13**, 399; cf. also Shen, J. Math. Physics **31**, 102–108 (1952); these Rev. **14**, 330; for the axially symmetric case, see Cabannes, Recherche Aéronautique no. **24**, 17–23 (1951); **27**, 7–16 (1952); these Rev. **13**, 597; **14**, 108; also Shen and Lin, NACA Tech. Note no. **2505** (1951); these Rev. **13**, 597]. The paper closes with a demonstration, under certain additional hypotheses, that the shock curve is asymptotically parabolic at infinity in both plane and axially symmetric flows. A different solution to this problem has been given by Whitham [Proc. Roy. Soc. London. Ser. A. **201**, 89–109 (1950); these Rev. **12**, 298].

D. Gilbarg (Bloomington, Ind.).

Morawetz, C. S., and Kolodner, I. I. On the non-existence of limiting lines in transonic flows. Comm. Pure Appl. Math. **6**, 97–102 (1953).

Les auteurs démontrent et complètent par des raisonnements très simples et très directs un résultat déjà donné par K. O. Friedrichs [mêmes Comm. **1**, 287–301 (1948); ces Rev. **10**, 638]. Les deux théorèmes qui font l'objet de cet article peuvent s'énoncer: I. Dans un écoulement continu autour d'un profil à courbure bornée, il ne peut apparaître de lignes limites lorsque croît le nombre de Mach à l'infini M tant que l'écoulement dépend continument de M et que la région supersonique est bornée. II. Si dans une famille d'écoulements autour de profils dépendant continument d'un paramètre, construits à partir du plan de l'hodographe, une ligne limite apparaît pour une certaine valeur critique du paramètre, le profil correspondant possède au moins un point en lequel la courbure est infinie. Ces théorèmes reposent sur un lemme relatif au signe du Jacobien de la transformation du plan physique dans le plan de l'hodographe; la démonstration exige seulement la continuité des dérivées secondes de la fonction de courant. *P. Germain.*

Davis, Julian L. On the nonexistence of four confluent shock waves. J. Aeronaut. Sci. **20**, 501–502 (1953).

Davies, H. J. The two-dimensional irrotational flow of a compressible fluid around a corner. Quart. J. Mech. Appl. Math. **6**, 71–80 (1953).

The author considers flows with stream functions $\psi = A\tau F(\tau)\sin 2\theta$ in terms of hodograph coordinates θ and $\tau = q^2/q^2_{\max}$, where A is a constant and $F(\tau)$ satisfies an hypergeometric differential equation. $F(\tau) = F(2.5, -3; 3; \tau)$ yields a flow in a corner discussed by J. Williams [Quart. J. Math., Oxford Ser. **20**, 129–134 (1949); these Rev. **11**, 222]. From $F(\tau) = \tau^{-2}F^*(0.5, -5; -1; \tau)$, where F^* is an independent solution of a type defined by Lighthill [Proc. Roy. Soc. London. Ser. A. **191**, 341–351 (1947); these Rev. **9**, 350; **11**, 870], the author obtains a flow which turns through a right angle and is stagnant at infinity. Just as in the case of Ringleb's flow around the edge of a semi-infinite flat plate [Z. Angew. Math. Mech. **20**, 185–198 (1940); these Rev. **2**, 169], this flow contains a locally supersonic region, and near the corner there are regions of triple-valued velocities. Four tables of $F(2.5, -3; 3; \tau)$, $F^*(0.5, -5; -1; \tau)$, and related functions are appended to the paper.

J. H. Giese (Havre de Grace, Md.).

Pistolesi, Enrico. Confronto fra due metodi di calcolo della portanza in corrente supersonica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 205–210 (1952).

This continues an earlier study of plane distributions of supersonic singularities [Aerotecnica **31**, 86–89 (1951);

these Rev. 13, 507]. A distribution of elementary vortices in a flow at Mach number M has the potential

$$\varphi_V(x, y, z) = \int_0^{x-\lambda|z|} d\xi \int_{\eta_1}^{\eta_2} f'(\xi, \eta)(x-\xi)z[(y-\eta)^2+z^2]^{-1}\Omega^{-1}d\eta$$

where $\Omega = \{(x-\xi)^2-(M^2-1)[(y-\eta)^2+z^2]\}^{\frac{1}{2}}$ and the integration extends over the part of the lifting surface in $z=0$ inside the Mach forecone of x, y, z. A doublet distribution has the potential $\varphi_D = \partial\varphi_S/\partial z$ where

$$\varphi_S = \int_0^{x-\lambda|z|} d\xi \int_{\eta_1}^{\eta_2} f(\xi, \eta)\Omega^{-1}d\eta.$$

The author shows that $\varphi_V = \varphi_D$ if $f' = \partial f/\partial\xi$ everywhere on the lifting surface and $f=0$ on those parts of the boundary not parallel to the y-axis. *J. H. Giese.*

Lavender, Robert E. A note on second-order supersonic flow theory. J. Aeronaut. Sci. 20, 435–437 (1953).

Nonweiler, T. The laminar boundary layer in slip flow. Coll. Aeronaut. Cranfield. Rep. no. 62, 57 pp. (2 plates) (1952).

Etude de la couche limite en admettant à la paroi une certaine vitesse de glissement et une discontinuité de température (hypothèses valables lorsque l'air est suffisamment raréfié, ou la couche suffisamment épaisse, pour que le libre parcours moyen des molécules entre en ligne de compte, tout en restant petit devant l'épaisseur de la couche). Le rapport de la vitesse de glissement à la vitesse en écoulement libre est de l'ordre de $R^{-1/2}$, où R est le nombre de Reynolds. En supposant que les propriétés de l'écoulement (contrainte de cisaillement, coefficient de passage de la chaleur, etc.) sont holomorphes au voisinage de la paroi, et en négligeant dans les développements les termes de l'ordre de $(R^{-1/2})^p$, $p>1$, on trouve à une certaine distance audessous de la paroi une surface fictive de glissement nul: sur cette surface, les conditions classiques aux limites sont valables, avec toutefois une température différente de celle de la paroi; de sorte que le problème peut être résolu, moyennant des hypothèses sur l'échange de chaleur.

Une discussion méticuleuse encore qu'un peu confuse montre que cette solution est correcte en première approximation lorsque le nombre de Reynolds est suffisamment élevé (par exemple lorsqu'il est de l'ordre de 100); elle serait alors sensiblement équivalente à celle donnée par Schaaf [Inst. Eng. Res., Univ. of California Rep. no. **HE-150-66** (1950)]. L'approximation ne peut être poussée plus loin par la même méthode.

Les valeurs de la vitesse de glissement et de la discontinuité de température sont finalement exprimées en fonction de deux paramètres, le "coefficient de transfert de quantité de mouvement" et le "coefficient d'accomodation", qui peuvent être déterminés expérimentalement.
M. Kiveliovitch (Paris).

Chiu, Wan-cheng. On the oscillations of the atmosphere. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. 5 (1952), 280–303 (1953).

The equations of motion of the atmosphere are set up under the assumptions that there is a zonal wind of constant angular velocity, that the temperature is a linear function of the height, and that the acceleration of gravity is constant. Motions of this kind are perturbed and the perturbation equations solved for wave-motions of an incompressible and homogeneous fluid. The periods of the waves of the first class are only slightly affected by the presence of the zonal wind; those of the second class are affected much more. For an autobarotropic atmosphere the presence of the zonal wind means that the equivalent depth of the atmosphere depends on the lapse rate of the lowest layer. The periods of waves of the first class are now as much affected by this lapse rate as by the presence of the zonal wind itself. These modifications to the period of the first class wave are within the range of tuning required by the resonance theory of atmospheric oscillations and are regarded as strengthening this theory. *G. C. McVittie* (Urbana, Ill.).

Van Mieghem, Jacques. Energy conversion in the atmosphere on the scale of the general circulation. Tellus 4 (1952), 334–351 (1953).

The "equations of balance" of the potential, kinetic, and internal energy are set up, by which are meant the equations obeyed by these quantities deduced from the equations of motions and of continuity. The energy fluxes and the rates of energy production and conversion are then deduced. If zonal averages are taken, the kinetic energy may be decomposed into kinetic energy of the mean motion and the mean kinetic energy of the large-scale eddies. The equations of balance for the kinetic energy of the zonal motion for that of the motion in meridional planes are set up. The definition of the large-scale diffusion of heat is deduced from the first law of thermodynamics and a tentative formulation of the second law is given. *G. C. McVittie.*

Queney, P. La résonance interne du jet-stream et son rôle dans la formation des cyclones. Proc. Indian Acad. Sci. Sect. A. 37, 213–222 (1953).

Plane flow of an incompressible fluid is considered in which the velocity of the basic flow is parallel to a straight line (the x-axis) and decreases linearly with distance from this line. The space rate of change of velocity is different on the two sides of the x-axis. This motion is perturbed under the limitation that the vorticity-component normal to the plane of motion is conserved. Solutions of the perturbation equations are obtained corresponding to wave-disturbances of the basic flow. If the shear changes again at a line parallel to the x-axis, a resonance phenomenon is predicted in which the perturbation increases in magnitude. When adapted to atmospheric motions, the theory can be interpreted as showing that a cyclonic wave is the resonance-wave produced in the jet stream by a local perturbation of the vertical component of vorticity. *G. C. McVittie* (Urbana, Ill.).

Štokman, V. B. Application of the method of complete flows for the computation of the circulation generated by a non-uniform wind in a sea of elliptical form. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. **1952**, no. 5, 57–68 (1952). (Russian)

L'auteur examine la distribution de courants complets dans deux cas de circulation s'établissant dans une mer de forme elliptique sous l'action d'un vent non uniforme. L'équation différentielle du problème a la forme bien connue dans la théorie de plaques élastiques chargées

$$\frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4} = -\frac{\mathrm{rot}_z\, T}{A},$$

en désignant par T la composante tangentielle de la pression du vent sur la surface de la mer, par x, y les coordonnées

horizontales, z la coordonnée verticale, par ψ la fonction du courant complet S, dont les composantes sont liées à ψ et à la vitesse horizontale (u, v) par les relations

$$S_x = -\partial\psi/\partial y = \int_0^h u dz, \quad S_y = \partial\psi/\partial x = \int_0^h v dz,$$

h étant la hauteur de la surface physique de la mer au-dessus du niveau $z=0$. On suppose qu'à ce niveau le courant est absent, et les tensions dues au frottement nulles. Autrement dit h est égal à l'épaisseur de la couche barocline de la mer. On suppose d'autre part que sur le contour de la mer la fonction ψ et sa dérivée normale au contour sont nulles. En admettant ensuite rot, T et A constants et en exprimant le contour par la formule $x^2/a^2+y^2/b^2=1$ on obtient une solution de l'équation différentielle

$$\psi = -\text{const.}\times[1-(x/a)^2-(y/b)^2]^2.$$

Dans un autre cas plus compliqué, avec le vent parallèle à l'axe y, en supposant sa pression tangentielle

$$T_y = T_0 + (T_a - T_0)a^{-2}x^2,$$

on obtient ψ sous la forme $\psi=\psi_0 x(1-x^2/a^2-y^2/b^2)^2$. D'après l'auteur ces formules permettraient d'expliquer les phénomènes observés sur la mer Noire et la mer Caspienne (circulation superficielle). Les formules se simplifient dans le cas de mers très allongées, où l'on peut, pour les sections centrales, supposer la mer équivalente à un canal infiniment allongé, fermé à l'infini. *V. A. Kostitzin* (Paris).

Štokman, V. B. The determination of the steady flows and of the density distribution in the normal midsection of a closed sea of elongated form. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. **1952**, no. 6, 57–72 (1952). (Russian)

En utilisant les résultats du mémoire analysé ci-dessus, l'auteur expose une méthode de calcul des courants complets dans la section transversale moyenne d'une mer fermée, allongée dans la direction du vent. En remplaçant dans les calculs la mer allongée par un canal rectiligne infini fermé à l'infini, on peut déterminer les courants complets et les variations de la densité quelle que soit l'irrégularité de la structure transversale du vent. L'auteur indique aussi le procédé permettant d'éliminer de ces calculs le coefficient de frottement turbulent latéral. *V. A. Kostitzin* (Paris).

Medwin, Herman, and Rudnick, Isadore. Surface and volume sources of vorticity in acoustic fields. J. Acoust. Soc. Amer. **25**, 538–540 (1952).

By a perturbation scheme the authors derive an approximate differential equation for the vorticity in small motions of a viscous compressible fluid, generalizing the result of Eckart [Physical Rev. (2) **73**, 68–76 (1948)]. They find that the "sources" (i.e., terms on the right-hand sides of the Poisson equation for the vorticity) in this approximation are unaffected by possible dependence of bulk viscosity on density. They name and discuss the relative importance of various terms. [The reviewer is unable to see why only one viscosity is allowed to depend on the density, why neither is allowed to depend on temperature, and why pressure is taken as a function of density only. On the basis of plausibility arguments of the type used by the authors it would seem that thermal dependences and the energy equation, not used by the authors, would have effects at least as large as those of the terms they discuss.] *C. Truesdell.*

Elasticity, Plasticity

Krettner, J. Elastostatische Grundformeln für allgemeine krummlinige Koordinaten. Österreich. Ing.-Arch. **7**, 11–21 (1953).

Textbook illustration of the use of tensor calculus [deriving ultimately from Ch. VI, §3 of G. Ricci and T. Levi-Civita, Math. Ann. **54**, 125–201 (1901)]. *C. Truesdell.*

Afendik, L. G. Some questions of the theory of finite deformations. Ukrain. Mat. Žurnal **3**, 98–117 (1951). (Russian)

The author derives the strain coefficients for finite deformations referred to the undeformed and deformed states, and relations between them. All the results appear to be well known. *L. M. Milne-Thomson* (Greenwich).

Conway, H. D. The stress distributions induced by concentrated loads acting in isotropic and orthotropic half planes. J. Appl. Mech. **20**, 82–86 (1953).

Following a previous paper [same J. **20**, 72–76 (1953); these Rev. **14**, 700], the author uses the Fourier integral method to obtain the solution for an isotropic half-plane subjected to a concentrated load at some distance from the straight edge. This result is shown to agree with that obtained by complex variable methods by Melan [Z. Angew. Math. Mech. **12**, 343–346 (1932)], and is then extended to solve the corresponding problem of the orthotropic half-plane. The two cases of the concentrated load applied along a line normal and parallel to the straight edge are treated. *R. M. Morris* (Cardiff).

Storchi, Edoardo. Le superficie eccezionali nella statica delle membrane. Rivista Mat. Univ. Parma **3**, 339–360 (1952).

Continuing his studies of the general solution of the equilibrium equations for the stresses on a curved surface [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 227–231 (1950); (8) **8**, 116–120, 326–331 (1950); these Rev. **11**, 556, 757; **12**, 219], the author takes up the question of whether by fortunate eliminations of arbitrary functions the fifth or possibly even lower order derivatives can be eliminated. For a general surface, he shows that such is not the case. *C. Truesdell* (Bloomington, Ind.).

Muštari, H. M., and Vinokurov, S. G. Determination of the stressed state for elastic equilibrium in the boundary zone of thin shells of certain types. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk **1**, 9–24 (1948). (Russian)

The authors present the application of Muštari's theory for finding the stresses in the boundary zone of shells [Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 129–136 (1948); these Rev. **9**, 547]. This paper begins with a summary of the general theory, introduces certain simplifications and gives an estimate of errors due to these simplifications. The theory is applied to the following shells: (a) shells of revolution with a spherical shell as a special case; (b) an elliptic cone. The construction of a stress function for boundary zones of shells is also presented. *T. Leser.*

Muštari, H. M. The theory of elastic equilibrium of plates and shells taking account of the initial stresses. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk **2**, 39–52 (1950). (Russian)

This paper is a continuation of the author's research on the theory of elastic equilibrium of shells with arbitrary

displacements [Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 121–134 (1949); these Rev. 11, 69]. The author begins with a brief summary of a previous (unavailable) paper [Trudy Kazan. Himiko-Tehnolog. Inst. 13 (1948)], where a prestressed shell is subject to two successive deformations. The author derives the equations of compatibility and those of equilibrium for shells as above. He modifies his general theory for special types of shells subject to certain kinds of deformations. He divides the shells into two types: (1) shells of small curvature, and (2) shells whose dimensionless curvature is of the order of one. The deformations are classified as follows: (a) the momentless deformation, where the bending deformation is small as compared with the deformation of the middle surface, (b) a mixed deformation, where deformations due to bending and of the middle surface are of the same order, (c) a bending deformation. The author discusses separately the most important cases which are combinations of the type of a shell and various kinds of initial, primary, and secondary deformations, before and at the loss of stability. He also presents simplified equations of neutral equilibrium, the determination of critical loads, discussing in particular shells of revolution. *T. Leser.*

Muštari, H. M. On the elastic equilibrium of a thin shell with initial irregularities in the form of the mean surface. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 743–750 (1951). (Russian)

The author considers a thin shell with initial irregularities of a rather special type, namely, rapidly damped creases or dimples on curves varying periodically with high frequency under loading which does not produce marked bending. The hypothesis permits the neglect of tangential displacements in the bending and curvature of the mean surface. The equations of equilibrium are set up and applied first to an oblate surface of revolution with a narrow zone, not too near the poles, of creases, the shell being acted upon by internal pressure. The resulting equations are specialised to the case of an oblate spheroid and an estimate of the critical pressure is obtained.

L. M. Milne-Thomson (Greenwich).

Alumyaè, N. A. On the critical value of the axially symmetric momentless stressed state of a long catenoidal shell. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 649–658 (1952). (Russian)

V. Z. Vlasov in his book, General theory of shells and its applications in technology [Gostehizdat, Moscow-Leningrad, 1949; these Rev. 11, 627] drew attention to a difference between shells of revolution with positive and negative Gaussian curvatures in their stressed state without bending (momentless). V. V. Novožilov [Theory of thin shells, Sudpromgiz, 1951] gave the theory of momentless stressed state of shells with negative Gaussian curvatures.

This paper begins with a short summary of the theory of stability for shells with negative curvatures, and makes a detailed study of the estimate of errors in the above theory. The equations for the local loss of stability of catenoidal shells with compressive loads on contours are derived and the critical load for a given catenoidal shell is evaluated. The solution shows that at the center of buckling the deformations of the middle surface of the shell are very small, which demonstrates again a considerable difference between the shells of positive and negative Gaussian curvatures.

T. Leser (Lexington, Ky.).

Stepanov, R. D., and Curkov, I. S. The computation of a cylindrical shell reinforced by a ring. Akad. Nauk SSSR. Inženernyĭ Sbornik **12**, 77–94 (1952). (Russian)

The authors consider a closed cylindrical shell fixed at one end and reinforced by a ring at the other one. The ring is made from a thin-walled rod with arbitrary rectangular cross-section, although the illustration shows a channel cross-section. The shell and the ring are under the action of arbitrary loads. The problem is solved in the following way: the ring and the shell are separated, and the influence of the shell on the ring is replaced by unknown deformations, and vice versa, the influence of the ring on the shell is replaced by the same unknown deformations. The condition of the continuity of deformations at the surface of contact requires that these deformations must be equal for the ring and for the shell. The authors write the differential equations of equilibrium for the ring and for the shell and the comparison of deformations at the surface of contact leads to a system of algebraic equations in four unknowns. The authors do not solve these equations, most of the operations are only indicated, but an outline of a procedure for numerical computations is sufficiently well presented. The solution of this problem is based on V. Z. Vlasov's theory of shells and the authors refer often to the following of his works: General theory of shells and its applications in technology, Gostehizdat, Moscow-Leningrad, 1949 [these Rev. 11, 627]; Thin walled elastic rods, Gosstroĭizdat, Moscow, 1940; Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 819–836 [these Rev. 11, 288]. *T. Leser.*

Oniašvili, O. D. On the theory of earthquake-proof sloping shells. Soobščeniya Akad. Nauk Gruzin. SSR **11**, 409–416 (1950). (Russian)

(Note: A sloping shell is an open shell of small curvatures.) This work is based on V. Z. Vlasov's theory given in his book, General theory of shells and its applications in technology [Gostehizdat, Moscow-Leningrad, 1949; these Rev. 11, 627]. The author considers shells which are rectangular when projected on a plane and are supported radially on all four edges. He limits himself to the following shapes: a shell of two different curvatures, a spherical shell, and a cylindrical shell; and each type is dealt with separately. The influence of an earthquake is reduced to a horizontal load acting on the entire surface of a shell. The author uses the momentless theory, and also the theory which takes bending into account. It is convenient to express the stresses in a form of double trigonometric series, when the supports are radial. The main task in this boundary-value problem is to determine the coefficients of these series. Although the paper devotes most of its space to the construction of the series and the determination of the coefficients, the author refers to a previous (unavailable) publication [Trudy Inst. Stroit. Dela Akad. Nauk Gruzin. SSR 3 (1950)] for a more detailed study of it. *T. Leser.*

Nowacki, W., and Turski, St. Application of the Fourier integral to the theory of orthotropic plates. Arch. Méc. Appl., Gdańsk **3**, 89–97 (1951). (Polish. Russian summary)

The author considers a composite plate, infinite in the y direction. The supports are parallel to the y-axis dividing the plate into parallel strips, each having different elastic properties. The plate is orthotropic which means that the elastic constants (elasticity modulus and Poisson ratio) are different in x and y directions. The loads act along a finite

line parallel to the y-axis and symmetrical with respect to the x-axis. The differential equation for plates of this kind was derived by T. Huber [La théorie générale des hourdis en béton armé, Lwow, 1914]. The author expresses the loads and the moments on the supports in the form of Fourier integrals; from the conditions of continuity on the supports he obtains a three-moment equation similar to the three-moment equation for continuous beams, and solves the problem, which means find the deflections. The total deflection is obtained by superposition of the deflection due to the load and the deflections due to the moments on the supports. A special case when one strip of an infinite plate is fixed at the edges and loaded by a concentrated force is obtained by substituting in general formulas the given conditions. It is compared with the solution of A. Nádai [Bauingenieur 2, 299–304 (1921), p. 301]. Nádai solved this particular problem in a different way and both solutions agree. *T. Leser* (Lexington, Ky.).

Kitover, K. A. On the use of special systems of biharmonic functions for the solution of some problems of the theory of elasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 739–748 (1952). (Russian)

The deflection of a loaded, thin plate of constant thickness and the stress-function in a problem of plane stress satisfy the biharmonic equation $\Delta^2\varphi=0$, combined with certain boundary-conditions. When the boundary-conditions are given on two parallel boundaries $(x=\pm a)$ of a plate the solution

$$\varphi=(A\cos kx+Bkx\sin kx+C\sin kx+Dkx\cos kx)e^{ky}$$

is assumed. The boundary-conditions lead to a transcendental equation for k and to three linear equations between A, B, C and D. Thus families of functions are calculated for several conditions on the boundaries $x=\pm a$. The same is done for a wedge-shaped and a ring-shaped plate, and for a state of plane stress in a circular cylinder with boundary-conditions on the bent surface. The transcendental equation for k, which has the general form $u_0+u_1\sin z+u_2\cos z=0$, where u_0, u_1 and u_2 are polynomials in z and z is a multiple of k, is solved by Newton's method. To solve a given problem of the theory of elasticity, a family of functions $\{\varphi_j\}$ is chosen, the members of which satisfy the boundary-conditions on a part of the boundary. Then an approximate solution of the problem is found by taking a finite series $\varphi=\sum_{j=1}^{n}c_j\varphi_j$ and determining the coefficients c_j in such a manner that the boundary conditions on the rest of the boundary are satisfied as well as possible.

W. H. Muller (Amsterdam).

Koltunov, M. A. The bending of rectangular plates taking account of large deflections. Akad. Nauk SSSR. Inženernyĭ Sbornik 13, 3–14 (1952). (Russian)

The author considers a flexible rectangular plate the edges of which do not necessarily remain straight under deformation by arbitrary transverse loading and forces applied in the middle surface. The method is to postulate expansions

$$\phi=\sum\sum A_{mn}U_m(x)V_n(y),\quad w=\sum\sum f_{mn}X_m(x)Y_n(y)$$

for the stress function and sag. The functions U, V, X, Y are chosen to satisfy individually the boundary conditions, and the coefficients A_{mn}, f_{mn} are then determined by a variational method. The author confines his actual calculations to the first approximation $m=n=1$, and applies the method to constant and uniformly varying load, to finite deflection, and to a plate bent by end loads applied at opposite edges.

L. M. Milne-Thomson (Greenwich).

Halilov, Z. I. Solution of the general problem of the deflection of a simply supported elastic plate. Amer. Math. Soc. Translation no. 87, 16 pp. (1953).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 405–414 (1950); these Rev. 12, 185.

Vaĭnberg, D. V. An analogy between problems on the plane stressed state and on the bending of a circular plate of variable thickness with unsymmetric loading. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 749–752 (1952). (Russian)

In this article it is shown, that the equation

$$D(\rho)\Delta\Delta\Phi+\frac{dD(\rho)}{d\rho}\left[2\frac{\partial^3\Phi}{\partial\rho^3}+\frac{2+\kappa}{\rho}\frac{\partial^2\Phi}{\partial\rho^2}-\frac{1}{\rho^2}\frac{\partial\Phi}{\partial\rho}+\frac{2}{\rho^2}\frac{\partial^3\Phi}{\partial\rho\partial\theta^2}-\frac{3}{\rho^3}\frac{\partial^2\Phi}{\partial\theta^2}\right]+\frac{d^2D(\rho)}{d^2\rho}\left[\frac{\partial^2\Phi}{\partial\rho^2}+\kappa\left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\theta^2}\right)\right]=\Psi(\rho,\theta)$$

can be interpreted in two ways: 1) as the equation for the deflection of a loaded, thin circular plate of variable thickness $h(\rho)$; then $D(\rho)=Eh^3/12(1-\nu^2)$, $E=$Young's modulus, $\nu=$Poisson's constant, $\kappa=\nu$, $\Phi(\rho,\theta)$ is the deflection; 2) as the equation for the stress function of a circular plate of variable thickness in a plane stressed state; then $D(\rho)=(Eh)^{-1}$, $\kappa=-\nu$, $\Phi(\rho,\theta)$ is the stress function. When the equation is solved with a certain function $D(\rho)$, the solution $\Phi(\rho,\theta)$ can be interpreted: 1) as the deflection of a circular plate, the thickness of which is given by $h=(12(1-\nu^2)E^{-1}D(\rho))^{1/3}$; 2) as the stress function of a circular plate, the thickness of which is given by $h=1/ED(\rho)$. *W. H. Muller.*

Kumar, S., and Joga Rao, C. V. Investigation of stresses around a hole in thin rotating disks of hyperbolic and parabolic profiles. J. Indian Inst. Sci. Sect. B. 35, 93–102 (1953).

For a variation of the thickness of the "thin" circular disk (see title) as r^n, where r is the distance from the axis and n positive or negative, the usual approximate two-dimensional treatment is used to discuss the stress-concentration at the circular hole. *E. Sternberg* (Chicago, Ill.).

✶Stippes, M. Large deflections of rectangular plates. Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 339–345. The American Society of Mechanical Engineers, New York, N. Y., 1952.

This investigation considers the problem of deflections of rectangular plates in which the membrane stresses are not neglected. The inclusion of the effects of these membrane stresses when considered in conjunction with usual bending stresses occurring in Lagrangian plate theory results in a set of two non-linear fourth-order partial differential equations known as von Kármán's equations for large deflections of plates [S. Timoshenko, Theory of plates and shells, McGraw-Hill, New York, 1940]. A procedure for obtaining the solutions of these equations is given for a class of boundary conditions. This group of boundary conditions is characterised by the property that no forces exist in the plane of the plate at the boundary. Under these circumstances, one can write the potential energy of the plate and the applied loads in a particularly simple fashion. Proper sets of functions are determined which satisfy the boundary conditions,

and the method of Ritz is used to obtain the solution. All of the plates considered are simply supported on two parallel edges. The remaining two parallel edges may be either simply supported, free or elastically supported. Maximum deflections are obtained as functions of the proper parameters for a square plate simply supported on all four sides loaded uniformly. As a by-product of the approach, one can also obtain the influence surfaces for a plate undergoing small deflections but still satisfying the above mentioned boundary conditions. (Author's abstract.)

R. M. Morris (Cardiff).

***Plantema, F. J., and van Alphen, W. J. Compressive buckling of sandwich plates having various edge conditions.** Anniversary Volume on Applied Mechanics dedicated to C. B. Biezeno, pp. 132–148. N. V. De Technische Uitgeverij H. Stam, Haarlem, 1953.

This paper is largely a refinement and extension of an earlier one by the first named author [Thesis, Delft, 1952; these Rev. **13**, 887]. The method of partial deflections developed in that paper is used to give a unified treatment for any set of boundary conditions. The effect of the bending stiffness of the facings, neglected in the earlier paper, is also considered. The investigation is restricted to long sandwich plates composed of isotropic materials. *H. W. March.*

Nowiński, Jerzy. On the theory of thin-walled beams with open cross-section under uniformly distributed load. Rozprawy Mat. **1**, 48 pp. (1952). (Polish. Russian and English summaries)

The author of this paper derived a theory of thin walled beams in a previous (unavailable) work [Biuletyn Inst. Techn. Lotn. **6** (1947)]. His theory differs from that given by Vlasov [Akad. Nauk SSSR. Prikl. Mat. Meh. **8**, 361–394 (1944); these Rev. **7**, 142], mainly in taking into account the so-called self-equilibrating stresses; Vlasov's theory neglects these stresses completely. In this paper the author presents the application of his theory to a particular case. The beam under consideration is a cylindrical shell cut lengthwise, which makes the cross-section open. The beam is fixed at one end and uniformly loaded. The assumptions are as follows: 1) the cross-sections remain plane; 2) the rigidity of the wall can be neglected; 3) the existence of a pseudo-plane state of stress in the wall; 4) the Poisson ratio equals zero. The author reduces the problem to a boundary-value problem with four unknown functions and solves it, considering bending and torsion separately. *T. Leser.*

Tekinalp, Bekir. Generalisation of the conjugate beam method to space rods. Bull. Tech. Univ. Istanbul **4** (1951), no. 1, 29–36 (1952). (Turkish summary)

The paper is concerned with the duality between the statics and the kinematics of curved rods. The relations developed in the paper agree with those given by the reviewer [An Introduction to the mathematical theory of structures, Mimeographed lecture notes, Brown Univ., Providence, R. I., 1947, p. 23 ff.]. *W. Prager.*

Olszak, Wacław. Sur la torsion nonlinéaire des barres anisotropes. Arch. Méc. Appl., Gdańsk **3**, 225–257 (1951). (Polish. French summary)

The author considers a bar of rectangular cross-section subject to pure torsion. The unit angle of twist is great but within the limits of elasticity. In such case the torsion is no longer linear and the shortening of the bar, the contraction of distances between cross-sections, and the longitudinal stresses cannot be neglected. The author starts with an outline of the general theory of non-linear torsion. The partial differential equations with second degree terms are difficult; therefore the author uses in his case a simplified theory suggested by C. Weber [M. S. G. Cullimore, Engineering structures (special supplement to Research), Academic Press, New York, 1949, pp. 153–164]. Isotropic and anisotropic bars are considered. The author compares numerical examples from his results with the experimental data given by Cullimore [loc. cit.], and the agreement is excellent. The linear theory of prismatic anisotropic bars based on stress function is appended. The bibliography mentions 21 books and papers on the subject.

T. Leser (Lexington, Ky.).

Pivovarov, A. M. Concentrations of shearing stresses in torsion of prismatic rods. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 253–259 (1953). (Russian)

The investigated prismatic rods are subject to pure torsion and their cross-sections are indented by reëntrant angles with sharp vertices. The choice of the torsion function depends on the point where the stress is to be determined. If the contour of the cross-section has no angles with sharp vertices, or if the point is distant from those vertices, the author refers to a previous (unavailable) paper [Akad. Nauk SSSR. Inženernyĭ Sbornik **15** (1953)] and to M. G. Slobodyanskiĭ [Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 245–250 (1951); these Rev. **13**, 288] where the convenient functions are listed.

In this work the author uses the method of finite differences to construct a stress function, which would determine the stresses in the neighbourhood of sharp vertices, where the concentration of stresses is unusually great. Assuming that the cross-section is symmmetrical and the axis of symmetry is through the vertex of an angle, the author derives general expressions for the stresses near the vertex. He illustrates his theory on three examples with the following cross-sections: (a) a square with a cut, where the angle is 0°; (b) an angular bar (the cross-section composed of two rectangles at a right angle); (c) a sector of a circle. The stresses near the vertex are tabulated and graphed.

T. Leser (Lexington, Ky.).

Sadowsky, M. A., and Sternberg, E. Pure bending of an incomplete torus. J. Appl. Mech. **20**, 215–226 (1953).

An exact solution in series form is presented for the stresses and displacements in an incomplete torus of circular centre line and circular cross-section under conditions of pure bending. The solution is preceded by a formal treatment of the case in which the shape of the cross-section is arbitrary. This more general problem is reduced to one of axisymmetry, which is in turn attacked by a modification of the stress-function approach of Timpe to rotationally symmetric problems in elasticity theory. The modified stress-function approach, which may be of interest beyond the present application, is referred to general orthogonal axisymmetric coordinates, and the solution of the specific problem considered here is based on the use of toroidal coordinates. The normal stresses acting on the circular cross-sections of the torus are evaluated numerically in an illustrative example and are compared with the results of previous approximate solutions of the same problem. The adaptation of the present solution to the determination of the initial stresses in a complete torus from which a wedge-shaped portion has been removed is indicated. (Author's summary.)

R. M. Morris (Cardiff).

Kupradze, V. D. The spatial dynamical problem of the theory of elasticity with given displacements on the boundary. Soobščeniya Akad. Nauk Gruzin. SSR. 10, 3–8 (1949). (Russian)

The present paper is an extension of a previous one [same Soobščeniya 9, 99–106 (1948); these Rev. 14, 336], and contains an essential simplification of the integral equations obtained previously for the boundary-value problem under consideration. This is based upon the integral equation

$$T(P,Q)=T(P,Q;0)+\frac{\omega^2}{4\pi}\int_B T(P,Q')T(Q',Q;0)d\tau_{Q'},$$

for the fundamental tensor $T(P,Q)\equiv T(P,Q;\omega)$, where ω is the frequency of vibration, which was introduced in the paper mentioned above. It is shown that the exterior problem with given displacements on the boundary and radiation condition at infinity always has a unique solution for any ω; there are no eigen-frequencies. *J. B. Diaz.*

Kupradze, V. D. The spatial dynamical problem of the theory of elasticity with given stresses on the boundary. Soobščeniya Akad. Nauk Gruzin. SSR. 10, 257–262 (1949). (Russian)

This paper is a continuation of the paper reviewed above and the one cited in that review. Let S be a simple closed smooth surface in 3 dimensions and B be its interior or exterior. The boundary-value problem under consideration consists in the determination of the displacement vector $\bar{u}=(u_1,u_2,u_3)$, continuous in $B+S$, with continuous second derivatives in B, satisfying the differential system

$$\Delta\bar{u}+\frac{\lambda+\mu}{\mu}\operatorname{grad}\operatorname{div}\bar{u}+k_2^2\bar{u}=0,\quad k_2^2=\frac{\sigma}{\mu}\omega^2,\quad \text{in } B,$$

and the boundary conditions

$$L_i\bar{u}\equiv\sum_{j=1}^{3}\tau_{ij}\cos(n,x_j)=f_i,\quad i=1,2,3,$$

where τ_{ij} are the components of the stress tensor and the f_i are given functions on S. The radiation condition at infinity is required when B is infinite. The author introduces the concept of an "antenna layer" potential and seeks a solution of the problem in the form of such a potential. He is led to the formulation of an equivalent system of Fredholm integral equations of the second kind, and to the result that if k_2 is not an eigen-vibration for the homogeneous interior boundary-value problem for the displacements, then the present problem has one and only one solution for arbitrary f_i. *J. B. Diaz* (College Park, Md.).

Šatašvili, S. H. On steady elastic vibrations with given displacements on the surface of the medium. Soobščeniya Akad. Nauk Gruzin. SSR. 10, 263–266 (1949). (Russian)

The problem of steady elastic vibrations of a plane elastic medium, given the displacements on the boundary, was considered by D. I. Šerman [Akad. Nauk SSSR. Prikl. Mat. Meh. 10, 617–622 (1946); these Rev. 8, 361] and I. N. Vekua [Doklady Akad. Nauk SSSR (N.S.) 60, 779–782 (1948); these Rev. 10, 87]. V. D. Kupradze [Soobščeniya Akad. Nauk Gruzin. SSR. 9, 99–106 (1948); these Rev. 14, 336; and the paper reviewed second above] gave the solution for bounded and unbounded three-dimensional bodies. In the present paper the author gives the solution for an elastic half space with given displacements on the surface of the medium. Writing the displacement vector as

$$(u,v,w)=\operatorname{grad}\Phi+\operatorname{curl}\psi,$$

one has

$$\Delta\Phi+k_1^2\Phi=0,\quad \Delta\psi+k_2^2\psi=0,\quad z>0,$$

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},\quad k_1^2=\frac{\lambda^2}{a^2},\quad k_2^2=\frac{\lambda^2}{b^2},$$

where a and b are the longitudinal (transversal) speeds of wave propagation, and λ is the frequency of vibration. Φ and ψ are to satisfy the boundary conditions (on $z=0$); $(u,v,w)(x,y,0)=(f_1,f_2,f_3)$, where the $f_i(x,y)$ are given functions. Following Šerman, the solution is sought in the form of integrals of certain particular solutions, and a system of Fredholm integral equations for the "densities" is obtained. *J. B. Diaz* (College Park, Md.).

Šatašvili, S. H. On steady vibrations of an elastic semi-space with given external forces. Soobščeniya Akad. Nauk Gruzin. SSR. 12, 265–268 (1951). (Russian)

The boundary-value problem here differs from the previous one in the paper reviewed above in that the boundary conditions are

$$aX_x+bX_y+cX_z=\mu f_1,$$
$$aY_x+bY_y+cY_z=\mu f_2,$$
$$aZ_x+bZ_y+cZ_z=\mu f_3,$$

where $a=\cos(x,n)$, $b=\cos(y,n)$, $c=\cos(z,n)$, n is the inner normal, $X_x,\cdots$ are the components of stress, μ is Lamé's constant, and the f_i are given functions. Following the procedure of D. I. Šerman [Akad. Nauk SSSR. Prikl. Mat. Meh. 10, 617–622 (1946); these Rev. 8, 361], by using integrals of certain particular solutions the boundary-value problem is reduced to an equivalent system of Fredholm integral equations. *J. B. Diaz* (College Park, Md.).

***Grammel, R. Nichtlineare Schwingungen mit unendlich vielen Freiheitsgraden.** Anniversary Volume on Applied Mechanics dedicated to C. B. Biezeno, pp. 108–118. N. V. De Technische Uitgeverij H. Stam, Haarlem, 1953.

The author considers the free elastic vibrations of materials satisfying a nonlinear relation between stress and strain of the type suggested by Kauderer [Ing.-Arch. 17, 450–480 (1949); these Rev. 12, 63]. The characteristic frequencies for the torsional vibrations of a circular cylinder clamped at one end and the bending vibrations of a prismatic bar simply supported at its two ends are found approximately as functions of a small parameter which is essentially the amplitude of the vibrations. A perturbation type approximation method is used in the analysis with this "amplitude" as perturbation parameter.

C. E. Langenhop (Ames, Iowa).

***Grammel, R. Oscillations non linéaires avec une infinité de degrés de liberté. Nichtlineare Schwingungen mit unendlich vielen Freiheitsgraden.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 45–58; discussion, p. 59, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953). (French and German)

This is a bilingual presentation of the paper reviewed above. *C. E. Langenhop* (Ames, Iowa).

Weidenhammer, F. Nichtlineare Biegeschwingungen des axial-pulsierend belasteten Stabes. Ing.-Arch. 20, 315–330 (1952).

A straight beam pinned at its ends and subjected to an axial pressure that is periodic in the time becomes unstable

with respect to lateral deflections when the frequency ω of the applied load is given by $2\omega_i/p$ ($p=1, 2, 3, \cdots$) when ω_i represents the frequency of any of the normal modes of free lateral oscillation of the beam. At these frequencies, the linear theory predicts infinite amplitudes when small disturbances are created in the unbent beam. If, however, the linear bending theory is replaced by a nonlinear theory which takes account of certain higher order terms in the lateral displacements, it is found that the amplitude of the oscillation is limited above even at the resonance frequencies, although the unbent state continues to be unstable. The author's principal object is to determine approximately these stable motions in the neighborhood of resonance. The author uses the following approximate method: The longitudinal displacement $u(x, t)$ and the lateral displacement $w(x, t)$ are written in the form $u=\sum_{i=1}^{n} Y_i(x) U_i(t)$, $w=\sum_{i=1}^{n} X_i(x) W_i(t)$. For $Y_i(x)$ and $X_i(x)$ the eigenfunctions of the linearized problem are taken, after which ordinary differential equations for $U_i(t)$ and $W_i(t)$ are found by using Hamilton's variational principle. If only one term is taken, for example (and a further reasonable approximation is made), the function $W(t)$ is found to satisfy the following equation

$$(*) \qquad \ddot{W}+(\alpha-\beta\cos\omega t)W+\gamma W^3=0,$$

i.e., the Mathieu equation with an additional nonlinear term that is cubic in W. (The quantities α, β, γ are constants.) The author considers first the existence of periodic solutions of (*) which have the same period or half the period of the given longitudinal pulsating load and shows that such oscillations do indeed exist when β and W are small; the method used is E. Schmidt's bifurcation theory for nonlinear integral equations. However, the author finds it more convenient to use a perturbation scheme in actually calculating the amplitudes of the periodic oscillations, and he carries out the calculations for quite a number of different types of oscillations. The stability of the periodic oscillations is investigated in the usual way by replacing W in (*) by $W+\delta W$, linearizing with respect to δW, and defining W as stable if all δW are bounded. An approximate treatment for the influence of viscous damping is given. *J. J. Stoker.*

Panovko, Ya. G. A method of direct linearization in nonlinear problems of the theory of elastic vibrations. Akad. Nauk SSSR. Inženernyĭ Sbornik **13**, 113–122 (1952). (Russian)

The author proposes to find the frequency of the free vibrations, of constant amplitude A, given by the equation $y''+f(y)=0$, where $f(y)$ is an odd function, by replacing $f(y)$ by ky, where k is a constant which minimizes $\int_{-A}^{A}\{[f(y)-ky]y\}^2dy$, the integral of the squared moment of the deviation of $f(y)$ from ky. He claims that the resulting first approximation to the frequency is more accurate than the first approximations by the method of Galerkin or Krylov-Bogolyubov. This claim is justified by numerical data but no mathematical proof is offered. The idea is also extended to forced vibrations, systems with several degrees of freedom, and the case of asymmetric restoring force $f(y)$.

L. M. Milne-Thomson (Greenwich).

Mapleton, Robert A. Diffraction patterns for solid delay lines. J. Acoust. Soc. Amer. **25**, 516–524 (1953).

Solutions of the vector equation of equilibrium for an isotropic solid are presented for the special case of harmonic time dependence and dissipationless wave propagation. The object is to investigate the angular distribution and speed of wave propagation of the components of the elastic wave field at large distances R from the source, which source region is a plane surface of finite extent. Solutions are obtained for specified discontinuities of stress on an interior plane of an unbounded solid, and prescribed stresses on the bounding plane of a semi-infinite solid. Several examples are considered in which the distribution of a single stress component is selected to simulate the stress produced by a high frequency quartz piezoelectric crystal. For the high frequencies of solid delay line operation and the isotropic solids commonly used, the angular distribution of the dominant components of the elastic wave field are essentially the same for the semi-infinite and unbounded solid, to terms in R^{-1}. The speed of propagation of the dominant field is characteristic of the type of source, shear or compressional, and the angular distribution is characteristic of the configuration of the source region. *A. E. Heins* (Copenhagen).

Kosminskaya, I. P. The interference of seismic waves generated by a harmonic source. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. **1952**, no. 4, 33–54 (1952). (Russian)

L'auteur examine l'interférence des ondes seismiques produites par une source harmonique. L'étude de la forme du hodographe de phases et de la courbe d'amplitudes en fonction de paramètres fondamentaux (vitesses et amplitudes des ondes composantes) permet d'établir des critères pour discerner dans les courbes observées la présence du phénomène d'interférence. L'auteur propose un procédé graphique pour construire la hodographe de phases et la courbe d'amplitude d'une onde harmonique composée quel que soit (théoriquement) le nombre des composantes.

V. A. Kostitzin (Paris).

Berti, Giuliana. Il problema del taglio nell'elasticità ereditaria. Rivista Mat. Univ. Parma **3**, 375–382 (1952).

By means of two linear integral operators, the author finds within Volterra's accumulative theory of elasticity an exact solution appropriate to bending of a prism by a transverse terminal load. This completes her earlier work on problems of St. Venant's class [Boll. Un. Mat. Ital. (3) **5**, 139–144 (1950); these Rev. **12**, 457]. *C. Truesdell.*

Thomas, T. Y. On the inclination of plastic slip bands in flat bars in tension tests. Proc. Nat. Acad. Sci. U. S. A. **39**, 257–265 (1953).

The author states that the usual explanation of the inclination of the slip bands in a wide tension specimen suffers from the following defects: 1) no plastic solution is exhibited and the boundary conditions on stress that must be satisfied over the surfaces of the slip band are neglected, and 2) as a consequence of this it is necessary to introduce the assumption, "extraneous to the basic equations of the theory" and "difficult to understand", that the extension in the direction of the oblique edges of the slip band vanishes. Using Hencky's stress-strain relations, the author then attempts to give a more rational explanation. Whereas it is usually assumed that the state of stress in the slip band is the same as in the rest of the specimen, the author admits stress continuities of the type discussed by this reviewer [Courant Anniversary Volume, Interscience, New York, 1948, pp. 289–300; these Rev. **10**, 82] at the parallel planes that bound the slip layer. The displacement components in the slip layer are assumed as linear functions of the coordinates, and the previously criticized condition of vanishing exten-

sion in the direction of the oblique edges of the slip band is incorporated at this stage. The inclination of the slip band is found to be 35°16′ under Mises' yield condition, and 35°16′ or 50°46′ under Tresca's yield condition. The displacement field obtained by the author is not continuous but involves a slip of particles along the bounding planes of the slip layer. This reviewer believes that this type of discontinuous displacement field is not admissible: a surface of discontinuity is but a mathematical idealization of a layer of small but finite thickness across which displacements vary rapidly in a continuous manner. The equations of plastic equilibrium must hold throughout this layer. Only those discontinuous displacement fields are admissible which can be obtained from continuous fields containing such layers by letting the thickness of each layer tend towards zero. The author's discontinuous displacement field is not of this type. Actually, the author's previous results on characteristic surfaces [J. Rational Mech. Anal. **1**, 343–357 (1952); these Rev. **14**, 113] indicate that slip bands cannot occur in a tension specimen under the exact equations of Mises' three-dimensional theory of plasticity. It is only when this theory is replaced by an approximate two-dimensional theory, purporting to describe the mechanical behavior of thin plastic sheets, that slip bands can occur in tension specimens [see, for instance, R. Hill, J. Mech. Phys. Solids **1**, 19–30 (1952); these Rev. **14**, 821].

W. Prager (Providence, R. I.).

Thomas, T. Y. The effect of compressibility on the inclination of plastic slip bands in flat bars. Proc. Nat. Acad. Sci. U. S. A. **39**, 266–273 (1953).

The author generalizes his previous analysis [see the paper reviewed above] by incorporating the effect of compressibility.

W. Prager (Providence, R. I.).

De Simoni, Franco. Su un particolare problema tridimensionale nella teoria della plasticità. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 623–634 (1951).

In the absence of body forces, the fundamental equations of the Mises theory of plasticity contain the following unknown functions of position: the mean normal pressure p, the components s_{ij} of the stress deviation, the velocity components v_i, and a positive factor of proportionality λ between the components of the velocity strain and the stress deviation. B. Finzi [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **76**, 222–238 (1941); these Rev. **7**, 501] derived a reduced system of equations containing only the stress deviation components s_{ij}. (Another reduced system containing only p and the velocity components v_i was given recently by T. Y. Thomas [J. Rational Mech. Anal. **1**, 343–357 (1952); these Rev. **14**, 113].) The present author remarks that, for the purpose of constructing particular solutions, it may be preferable to use a less reduced system of equations of lower order. As an example, he uses equations involving p, λ, and v_i to study solutions for which the velocity components v_i depend in an exponential manner on a linear form of the coordinates x_1, x_2, and x_3. He finds that the stress field is uniform in this case. (The author seems to be unaware of the fact that he has not actually found a three-dimensional flow. When the flow field is referred to the principal axes of the uniform stress field, it is seen to be a field of plane flow.)

W. Prager (Providence, R. I.).

***Edelman, F. On the coincidence of plasticity solutions obtained with incremental and deformation theories.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 493–498. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author defines radial loading to be such that the existing stress tensor is proportional to the stress tensor at some previous time. First, the author shows that the proportionality factor is independent of position unless one of the principal stresses (at the previous time) vanishes. The latter cases are omitted from the remainder of the discussion which generalizes a theorem of A. A. Ilyushin [Akad. Nauk SSSR. Prikl. Mat. Meh. **10**, 347–356 (1946) = Graduate Division of Appl. Math. Brown Univ., Translation no. **RMB-17** (1947); these Rev. **8**, 240]. Two theorems are proved. First, for a general class of loading functions and power type of stress-strain relation, it is shown that for proper boundary conditions radial loading must exist. This result generalizes the theorem of Ilyushin. Then, the converse theorem is demonstrated. (Since the existing stress tensor is time dependent, the author is implicitly assuming that the inertia terms are of lower order than the stress derivatives.)

N. Coburn (Ann Arbor, Mich.).

Green, A. P. The plastic yielding of notched bars due to bending. Quart. J. Mech. Appl. Math. **6**, 223–239 (1953).

Limit analysis is applied to the problem of estimating the yield point couples of notched bars bent under conditions of plane strain. Deep notches with circular or V-shaped roots and shallow notches of arbitrary shape are considered. Constraint factors are worked out for a bar with a single notch as well as a bar with two symmetrical notches. For bars with deep V-shaped notches two types of slip line field are described. The first corresponds to large notch angles and resembles the slip line field in a notched tension specimen. The second type of slip line field corresponds to small notch angles and involves a pair of circular slip lines bounding a rigid pivot about which the rigid ends of the bar rotate. Experiments on wide, deeply notched bars are described. The yield point couples, the surface deformation, and the regions of plastic deformation were found to agree well with the theory.

W. Prager (Providence, R. I.).

***Heyman, J., and Nachbar, W. Approximate methods in the limit design of structures.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 551–560. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The paper discusses the collapse of a general plane frame with rigid joints, the frame being subjected to concentrated or distributed loads which act in the plane of the frame. Limit moments are assumed unaffected by the presence of axial and transverse forces. The analysis stems from previous work on the limit design of beams and frames [Greenberg and Prager, Proc. Amer. Soc. Civil Engrs. **77**, no. 59 (1951)], and techniques are devised for the determination and the progressive refinement of lower and upper bounds to the collapse loads. Specific examples are treated in detail.

H. G. Hopkins (Providence, R. I.).

Salvadori, M. G., and DiMaggio, F. On the development of plastic hinges in rigid-plastic beams. Quart. Appl. Math. 11, 223–230 (1953).

The paper discusses the development of plastic hinges in a free-ended uniform plastic-rigid beam in motion under the action of a distributed transverse load. This load is symmetrical about the midpoint of the beam, increases in time at the same rate for all points, and, in any particular case, the type of distribution always lies between the extreme cases of a uniformly distributed load and a concentrated central load, the transition from the former to the latter extreme being represented by the variation of a parameter c from zero to infinity. As the total applied load increases uniformly, the motion is characterized by the passage through certain distinct phases. In the first phase, there is rigid translational motion, and, in the second phase, there is a central hinge. The nature of the third phase depends upon whether $c>c_1$ or $c<c_1$, c_1 a certain particular value; if $c>c_1$, then two lateral symmetrical hinges develop and move inwards to the centre, but, if $c<c_1$, then the central hinge divides and two symmetrical hinges move outwards from the centre. The nature of the motion is not further discussed, and, in particular, there is no discussion of the important problem of unloading. The results for the concentrated load case agree with those found previously by Lee and Symonds [J. Appl. Mech. 19, 308–314 (1952)].

H. G. Hopkins (Providence, R. I.).

MATHEMATICAL PHYSICS

★**Destouches, Jean-Louis. Méthodologie. Notions géométriques.** Traité de physique théorique et de physique mathématique, tome I. Gauthier-Villars, Paris, 1953. xiv+228 pp. 3,000 francs.

This book, which is the first volume of a projected series of monographs on the foundations of physics, falls essentially into two parts.

The first part (pp. 1–69) constitutes a discussion of the various conditions which, the author holds, must be satisfied by a satisfactory formalization of a branch of physics. The reviewer finds the formulation of most of these conditions obscure, and does not understand the arguments advanced to show that they are necessary. The discussion would be easier to follow if the author had provided some specific examples of axiomatizations of branches of physics: we may hope that such examples will be supplied in later volumes of the series.

The remainder of the book is devoted to a presentation of various geometries (projective, affine, and euclidean), and to a brief treatment of some notions connected with mass (center of mass, in particular, and ellipsoid of inertia); here much emphasis is put on the lattice-theoretic properties of the set of linear subspaces of a given space. Since, as the author himself says on page 71, "les démonstrations sont omises et l'exposé ne prétend être ni complet ni rigoreux", it is not clear why these chapters could not have been replaced by a set of references to standard works on geometry.

Parts of the book were originally given as lectures at the Sorbonne, and the various chapters end with "Exercises", some of which seem unconscionably difficult: thus Exercise 1, on page 69, reads: "Axiomatiser diverses théories physiques; faire des démonstrations d'independance des concepts et des postulats". It would have seemed more reasonable to see this project proposed as a life work, rather than as one of seven "Exercises".

J. C. C. McKinsey.

Pi Calleja, Pedro. On regularity and conventions in the concept of physical magnitude. Math. Notae 12–13, 19–31 (1952). (Spanish)

A discussion of the logical difficulties of dimensional analysis. It is pointed out that the difficulties are not mathematical so much as in agreeing on definitions (conventions).

G. Birkhoff (Cambridge, Mass.).

Optics, Electromagnetic Theory

Keller, Joseph B. Parallel reflection of light by plane mirrors. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. EM-36, 6 pp. (1951).

Keller, Joseph B. Parallel reflection of light by plane mirrors. Quart. Appl. Math. 11, 216–219 (1953).

A number of plane mirrors have a common point A. The planes of the mirrors cut a sphere with centre A in a spherical polygon, the reflecting surfaces of the mirrors pointing into the interior of the polygon. The problem is to find all configurations of the mirrors (i.e., all forms of the spherical polygon) such that every ray incident on a reflecting surface of the set of mirrors is ultimately reflected back along the direction from which it came. The author proves that the number of mirrors cannot exceed three, and that, in the case of three mirrors, the angles of the spherical triangle must be $(\pi/2, \pi/3, \pi/4)$, $(\pi/2, \pi/3, \pi/5)$ or $(\pi/2, \pi/2, \pi/n)$ with n even. For $n=2$ this last case gives the well-known reflecting corner formed by three mutually perpendicular mirrors.

J. L. Synge (Dublin).

Lansraux, Guy. Diffraction instrumentale. Rev. Optique 31, 321–333, 444–456, 545–560 (1952); 32, 73–90, 213–225 (1953).

The effect of diffraction in the formation of optical images is considered first for classical instruments and then examined under the assumption of local variations in the transmittance of the optical components. The concept of "amplitude filters" is introduced and possibilities are considered for using these to improve resolving power or contrast. The mathematical technique is described for applying the theory to treat various practical aspects of image formation. Special attention is given to the center of the diffraction pattern and its approximate determination by a simple formula in the case of small aberrations. For the remote area of the image, the theory gives the asymptotic expression for the intensity along the radial direction of the diffraction pattern. In considering various methods for evaluating images, the author proposes the "degree of modulated intensity" as a kind of general test of the quality of the diffraction pattern.

E. W. Marchand (Rochester, N. Y.).

Hopkins, H. H. On the diffraction theory of optical images. Proc. Roy. Soc. London. Ser. A. 217, 408–432 (1953).

The phase-coherence function introduced by Hopkins [same Proc. 208, 263–277 (1951); these Rev. 13, 407] is

used to formulate a theory of image formation involving the diffraction distribution function of the optical system and a function giving the structure of the object. A very general result is obtained as a quadruple integral involving the above functions and their complex conjugates. The integral is evaluated by means of Fourier transforms which suggest introducing an "effective source." Application to special cases is found to reproduce known results. Among other examples considered, the theory is used to investigate the effects of various modes of illumination on the images of simple periodic structures. *E. W. Marchand.*

Jones, D. S. Diffraction by a thick semi-infinite plate. Proc. Roy. Soc. London. Ser. A. **217**, 153–175 (1953).

The form of the exact solution for the diffraction of a two-dimensional plane harmonic wave by a semi-infinite plate of thickness d is found. The solution involves constants which satisfy an infinite set of equations, and these equations are solved when d is small compared with the wave-length. It is shown that, in the neighbourhood of the shadow, the field is that of a single semi-infinite plane occupying the nearer face of the plate, whatever d, if terms of $O(R^{-\frac{1}{2}})$ are neglected, R being the distance of the point of observation from the edge. It is further shown that, when d is less than wave-length/10, the plate behaves as a semi-infinite wave-guide whose sides project beyond the end of the plate by an amount $0.11d$ together with, when the plane of polarization of the incident wave is perpendicular to the plate, a two-dimensional magnetic dipole at the end of the guide. When terms of $O(kd)$ can be neglected, it appears from this result and Hanson's [Philos. Trans. Roy. Soc. London. Ser. A. **229**, 87–124 (1930)] work on a plate with a cycloidal end that the exact shape of the end of the plate is of no importance; the plate behaves as a semi-infinite wave-guide.

The extension of the theory to the diffraction by a thick plate of finite length is briefly discussed. The theory is also extended to incident scalar waves whose direction of propagation does not lie in the plane perpendicular to the plate and, from this, the field due to an incident electromagnetic wave is deduced. It is found that, for all values of d, the diffracted electromagnetic wave at any point is effectively traveling along a cone of semi-angle θ_0 and axis the nearer edge, where θ_0 is the angle between the edge and the direction of propagation of the incident wave. When d is small compared with the wave-length, the plate acts as two parallel planes together with a line of magnetic dipoles at the end of the planes. (From the author's summary.)

A. E. Heins (Copenhagen).

Müller, Claus. Randwertprobleme der Theorie elektromagnetischer Schwingungen. Math. Z. **56**, 261–270 (1952).

The author presents a uniqueness and existence proof for the interior and exterior problems for Maxwell's equations, the former, of course, under suitable restrictions as to eigenvalues. The method makes use of the "capacity" matrix and a mapping similar to that employed by H. Weyl in the scalar problem [Math. Z. **55**, 187–198 (1952); these Rev. **14**, 225]. A Green's tensor is, however, not used, which avoids the complication encountered by the reviewer in verifying the fulfillment of the boundary conditions [Proc. Nat. Acad. Sci. U. S. A. **38**, 342–348 (1952); these Rev. **14**, 823]. The uniqueness part of the demonstration is made to depend on a previous paper of the author [Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1945/46**, no. 3 (1950); these Rev. **12**, 305]. *W. K. Saunders.*

***Ollendorff, Franz. Berechnung magnetischer Felder.** Springer-Verlag, Wien, 1952. x+432 pp. $15.70.

This book represents the largest collection of analytical detailed solutions of magnetic field problems. Because of the greater importance of magnetic fields in electric power devices and machines, most applications pertain to the power field, though several applications to measuring instruments are also included. For the sake of completeness, simpler solutions, given in many other publications, are repeated here. In general, the method of presentation follows closely the earlier work of the same author: Potentialfelder der Elektrotechnik [Springer, Berlin, 1932].

There are four chapters dealing, respectively, with solutions of scalar potential problems in terms of real function systems and by means of functions of a complex variable, with solutions of vector potential fields, and with electrodynamic forces. The first chapter gives a brief survey of Laplace's differential equation in various coordinate systems and of the spherical Legendre function system. Applications are shown to the evaluation of geomagnetic fields, using also three-dimensional inversion; to bar magnets; and to magnetic field distributions in salient poles and slotted armatures of electrical machines. Care is taken to arrive at factors or characteristic numbers valuable for design purposes. The chapter concludes with the scalar potential solution of a circular current loop, of the axial magnetic field of a shielded deflection coil, and of the fields surrounding the end connections of machine windings. The formulation of each problem and its reduction to solvable basic elements is admirably demonstrated.

The second chapter reviews first the concept of conjugate functions and the elements of conformal mapping for two-dimensional field problems. Applications are shown to periodic structures as approximated by elliptic functions; and to the mapping of rectangular regions encountered in slots, pole shoes, interpoles, and transformer windows. In each case, the solution is carried into considerable detail from the formulation of the physical problem, through the mathematical, idealized expression, to the evaluation of significant quantities illustrated by graphs of the results.

The third chapter reviews the concept of vector potential and inductance, gives the simpler results for parallel round wires, and then shows more advanced applications to transformer problems, particularly leakage field distributions. After establishing the general field distribution of an axially symmetrical current flow, detailed solutions are given for the current in collector rings of squirrel-cage rotors and their inductances.

The fourth and final chapter deals with magneto-mechanical force actions in electromagnets, bus-bar systems, machines, and instrument movements. Particularly the last-named applications are for the most part original contributions.

The book contains 287 figures, many giving schematic reproductions of the actual physical arrangements so as to facilitate the recognition of the most efficient mathematical formulation. Wherever desirable in the interest of practical interpretation of the results, the author uses approximations to the rigorous solutions and in several instances calls upon graphical methods. The clarity of conception as well as of presentation are exceptional, and well supported by the attractive format of the book. *E. Weber.*

Davy, N., and Langton, N. H. The external magnetic field of a single thick semi-infinite parallel plate terminated by a convex semi-circular cylinder. Quart. J. Mech. Appl. Math. 6, 115–121 (1953).

Assuming the surface of the plane slab to have constant magnetostatic potential value V_0, the authors transform the boundary into the real axis of a W-plane by a method originally due to Schwarz [J. Reine Angew. Math. 75, 292–335 (1873)] and also used by Daymond and Hodgkinson [Quart. J. Math., Oxford Ser. 10, 136–144 (1939)] (to whom the transformation function is attributed). The magnetic field vector is computed numerically along the surface of the slab as well as in the plane of symmetry. *E. Weber.*

Rauch, S. E. Cycloidal motion of electrons. Math. Mag. 26, 255–262 (1953).

Payne, W. T. Spinor theory of four-terminal networks. J. Math. Physics 32, 19–33 (1953).

The network transformation $V'=aV+bI$, $I'=cV+dI$, here considered without the usual restriction $ad-bc=1$, is discussed in terms of a geometrical representation of a spinor by a vector in ordinary space, with an associated angle [W. T. Payne, Amer. J. Phys. 20, 253–262 (1952); these Rev. 13, 871]. *J. L. Synge* (Dublin).

Percival, W. S. The solution of passive electrical networks by means of mathematical trees. Proc. Inst. Elec. Engrs. Part III. 100, 143–150 (1953).

From the graph of a network, homogeneous forms called linkages are constructed which obey rules similar to the laws of Kirchhoff. For a network with no inductive or triode couplings it is shown that the transfer admittances of the network are the quotients of the tree form and the corresponding linkages when the indeterminates are replaced by the associated branch impedances. Thus instead of calculating the transfer admittance by the expansion of determinants, the expanded determinants required can be written directly from the graph of the given network. A similar theorem also based on graph-theoretic methods is given for network impedances. *C. Saltzer* (Cleveland, Ohio).

Haacke, Wolfhart. Ein Stabilitätskriterium für Schwingungen in n-fachen Netzen mit pulsierenden Parametern. Arch. Elektr. Übertragung 6, 515–519 (1952).

The author studies the stability of a linear electric circuit of n meshes whose inductances and capacitances perform small oscillations of circular frequency α about constant mean values. The resistances are considered negligible. By suitable transformations the differential equations of the circuit can be reduced to the form

$$\alpha^2 d^2v_j/du^2+\alpha_j^2 v_j+\tau\sum_{k=1}^{n} p_{jk}(u)v_k=0 \quad (j=1,\cdots,n),$$

where the $p_{jk}(u)$ have period 2π and mean value zero, and the parameter τ is small. To this system the stability criteria developed by the author are applied [Math. Z. 57, 34–45 (1952); these Rev. 14, 646]. They lead to certain curves near $\tau=0$ in the (α,τ)-plane separating regions for which all solutions $v_j(u)$ of the differential equation are bounded for real u from regions for which unbounded solutions occur. The case $n=2$ is studied in more detail under the special assumption that $p_{jk}(u)=a_{jk}\cos u$, with constant a_{jk}.

W. Wasow (Los Angeles, Calif.).

Schwetman, Herbert D., and Brown, S. Leroy. The application of the Laplace transformation and a mechanical harmonic synthesizer in the analysis of electric circuits. Rev. Sci. Instruments 24, 375–379 (1953).

***Kalin, Theodore A. Formal logic and switching circuits.** Proceedings of the Association for Computing Machinery, Pittsburgh, 1952, pp. 251–257. Richard Rimbach Associates, Pittsburgh, Pa., 1952.

Craven, T. L. Logic and the circuit designer. Electronic Engrg. 25, 257–259 (1953).

Ikehara, Shikao. A method of Wiener in a nonlinear circuit. Research Laboratory of Electronics· Massachusetts Institute of Technology, Cambridge, Mass. Tech. Rep. No. 217 (1951). 22 pp.

This report gives an expository account of a method due to Wiener [Radiation Laboratory, Mass. Inst. Tech., Rep. no. 129 (1942)]. His method is to solve for the voltage across a nonlinear device in terms of the random voltage and then to get statistical averages on the assumption that the current-voltage function of the nonlinear element and the system transfer function are given.

From the author's abstract.

Quantum Mechanics

***Sokolov, A., i Ivanenko, D. Kvantovaya teoriya polya. (Izbrannye voprosy.) [The quantum theory of fields. (Selected questions.)]** Gosudarstv. Izdat. Tehn.-Teor. Lit. Moscow-Leningrad, 1952. 780 pp. 14.30 rubles.

The book consists of two separate parts. Part I, "Quantum electrodynamics" by A. Sokolov (pp. 9–480), is based on a lecture course for students specializing in theoretical physics. Particular consideration is given to papers published in Russia. Knowledge of the authors' "Classical theory of fields" [2nd ed., Gostehizdat, Moscow-Leningrad, 1951; see these Rev. 13, 95 for a review of the 1st ed.] is presupposed. The quantum theory of the electron and of the electromagnetic field is presented in the way customary before the use of the interaction representation. The quantum theory of radiation is developed in considerable detail and applied to a number of problems. (This section of the book is about equivalent to Heitler's "Quantum theory of radiation" [2nd ed., Oxford, 1944].) The theory of positronium and of cosmic radiation processes is outlined. A chapter on the theory of the vacuum covers topics hardly accessible elsewhere in textbooks, like self-energies, regularization methods, radiative corrections, Lamb-shift, infrared catastrophe, etc. There is also an interesting paragraph on the domain of applicability of quantum theory where the Soviet point of view of the interpretation of quantum mechanics is presented. We read, for instance, on p. 184: "According to the concepts of dialectic materialism, quantum mechanics should, first of all, reflect objective regularities of the 'microworld' which should not depend upon the particular type of instrument used to investigate them. Bohr's 'Principle of Complementarity' appears to be a perverse idealistic assumption according to which it would be impossible to explain in a consistent way the properties of the 'microworld'." And on p. 192: "Summarizing, we can say that quantum theory describes correctly many regu-

larities of simpler types of motion. Unfortunately, bourgeois scientists try to litter the new theory with a series of reactionary assumptions which handicap considerably the progress of science and lead the scientist away from the correct investigation of the phenomena of nature. Only from the point of view of dialectic materialism will it be possible to uncover the idealistic deviations . . ."

Part II "Introduction into the theory of the elementary particles" by D. Ivanenko (pp. 481–780) is a survey of the present state of the theory and an attempt to analyze its difficulties which probably many Western readers would find interesting. There is no particular stress on Soviet views and achievements. The relevant aspects of relativistic quantum mechanics, interaction representation, S-matrix methods, and some of Feynman's methods are discussed. The main attempts to generalize the quantum theory of fields and particles are classified into three groups: A) hypotheses based on the concept of a fundamental length (quantization of space and time, non-linear, non-local and reciprocal theories); B) hypotheses based on the use of compensating fields (higher order derivatives, regularization methods); C) composite structure of elementary particles (de Broglie's method of fusion, Fermi-Yang's theory of π-meson decay). Possible links between the elementary particles and the gravitational field are considered; the attempts to construct quantum theories of the gravitational field are described in some detail. A survey is given of the theories of multiple processes (β-decay, meson production, multiple processes, and interaction forces), of the isotopic shift effects, and of the interaction between neutron and electron. *E. Gora* (Providence, R. I.).

v. Krbek. Anfangsgründe der Quantenmechanik. Wissensch. Z. Univ. Greifswald. Math.-Nat. Reihe 1, no. 1, 14–27 (1952).

Expository paper.

Falk, Gottfried. Eine kanonische Formulierung der Relativitätsmechanik und ihr quantentheoretisches Analogon. Z. Physik **132**, 44–53 (1952).

An invariant canonical formulation of classical relativity mechanics of a particle is given. The four space-time coordinates are taken as functions of a parameter, which it is convenient to choose as the arc length of the world line of the particle. Conjugate momenta are introduced. It is found that the Hamiltonian function is the rest mass of the particle. To go over to quantum mechanics, commutation relations among the coordinates and momenta are introduced, and by replacing the momenta by differential operators one gets a Schrödinger equation for the square of the rest mass. Discrete values for this quantity will be obtained only if the corresponding operator does not commute with any of the four momenta. It follows that the rest mass will have a continuous spectrum in the case of any static field, and also that, if the rest mass has a discrete value, there will be an uncertainty in the energy. *N. Rosen* (Haifa).

McShane, E. J. The spectrum of the harmonic oscillator. Virginia J. Sci. (N.S.) **4**, 7–10 (1953).

The usual eigenvalues of the harmonic oscillator, and the fact that there are no others, are derived from the well-known algebraic relations between the momentum, position, and energy operators. No special representation of states and observables is needed. Properties of the projection operators involved in the spectral decomposition of the energy operator are used. *T. E. Hull.*

Ascoli, R. Interazioni non localizzabili. Confronto fra varie formulazioni. Nuovo Cimento (9) **10**, 745–753 (1953).

The non-local field theories discussed by Bopp, Uhlenbeck, and Pais, Peierls and McManus, and Wataghin are derived as particular cases of the non-local Lagrangian in which all three field operators in the interaction term are taken at different points. *H. C. Corben.*

Synge, J. L. Primitive quantization in the relativistic two-body problem. Physical Rev. (2) **89**, 467–471 (1953).

The paper deals with the relativistically invariant, Bohr-Sommerfeld type of quantization of a two-particle system. The two-particle system is described by means of an eight-dimensional space-time, with a separate time coordinate for each particle. For the case of a hydrogen-like atom, the interaction is taken in the form of the Coulomb potential, but with the Euclidean distance replaced by the Minkowski interval. Hamilton's optical method [J. L. Synge, Hamilton's method in geometrical optics, Inst. for Fluid Dynamics and Appl. Math., Univ. of Maryland, 1951; these Rev. **13**, 706] is used. The Hamiltonian equation is set up in terms of the proper masses of the particles, their momentum four-vectors, and their interaction. By means of this the rays for the system are obtained. De Broglie waves are then defined, and quantization comes about as a resonance condition on these waves. This leads to an expression for the energy of the system (for arbitrary masses and velocities) which yields the correct approximate energy levels of a hydrogen-like atom. *N. Rosen* (Haifa).

De Sloovere, H. Sur la stabilité, au sens de Th. De Donder, des lois de la mécanique ondulatoire. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 336–339 (1953).

Burton, W. K., and Touschek, B. F. Commutation relations in Lagrangian quantum mechanics. Philos. Mag. (7) **44**, 161–168 (1953).

Schwinger's method [Physical Rev. (2) **82**, 914–927 (1951); these Rev. **13**, 520] for deriving commutation relations for field variables is applied to two examples for which the equations of motion derived from the variation principle differ from those derived by evaluating the time derivatives as commutators with the energy operator H and applying Schwinger's canonical commutation rules. A method based on work by Peierls [Proc. Roy. Soc. London. Ser. A. **214**, 143–157 (1952), pp. 143–145; these Rev. **14**, 520] leads to correct commutation relations for the problems considered. Similar considerations are extended to a more general type of Lagrangian. *C. Strachan* (Aberdeen).

Melvin, M. Avramy. The new classical electrodynamics. Nature **171**, 890–892 (1953).

Muto, Toshinosuke, and Inoue, Kenzo. Notes on Dirac's new quantization method in the field theory. Progress Theoret. Physics **5**, 1033–1044 (1950); errata: **6**, 445 (1951).

It might be expected that Dirac's new method of field quantisation [Dirac, Proc. Roy. Soc. London. Ser. A. **180**, 1–40 (1942); these Rev. **5**, 277; Pauli, Rev. Modern Physics **15**, 175–207 (1943)] would give a vanishing electron self-energy, to any order of approximation, in the one-electron theory. Here the fourth order self-energy of a free-electron and the Lamb-Retherford shift (second-order radiative reaction for electron bound in a hydrogen atom) are calculated

using Dirac's method. The former is found to diverge. The remedy provided by Dirac's method is also too drastic for the second-order electromagnetic interaction with a free or bound electron: the computed Lamb-Retherford shift vanishes. The errata occur as misprints in equations, not as parts of the text. *C. Strachan* (Aberdeen).

Loinger, A. Un esempio elettrodinamico di teorema di ortogonalità alla Van Hove. Nuovo Cimento (9) **10**, 498–499 (1953).

As was shown by the reviewer [Physica **18**, 145–159 (1952); these Rev. **14**, 118], for a neutral scalar meson field in interaction with static point sources, the Hilbert space spanned by the eigenstates of the field is orthogonal to the corresponding space in absence of interaction. The author derives a similar result for the electromagnetic field in interaction with a nonrelativistic electron, in the dipole approximation and neglecting in the hamiltonian the term containing the square of the potential. *L. Van Hove.*

Green, H. S. First-order meson wave equations. Physical Rev. (2) **89**, 965–967 (1953).

The mass of the ξ-meson is almost exactly twice that of the π-meson. A practical method of solution of various first order wave equations is illustrated first by application to Kemmer's equation for mesons of spin 0 and 1. This is then given a simple generalisation which leads to field equations corresponding to particles of which one kind has twice the rest mass of the other. Difficulties arise in the sign of the energy. A further generalisation gives possible mass values n/n_1 times a basic value where $n_1=1, 2, \cdots, n$. The sign of the energy of the corresponding fields alternates with decreasing mass values. About one-half of the mass values have to be rejected as inadmissible. The theory can be adapted for particles of half-odd-integral spins. *C. Strachan* (Aberdeen).

Green, H. S. A generalized method of field quantization. Physical Rev. (2) **90**, 270–273 (1953).

In the endeavour to find a means of relaxation of the present rigid structure of field theory a generalization of field quantisation is investigated of which Bose and Fermi statistics are the simplest examples. For spin-half fields the scheme of quantisation is arranged to secure that $\partial_a\Psi(x)=i[P_a, \Psi(x)]$, where the energy-momentum vector is defined by $P^a=\sum_r p_r^a[b_r^*, b_r]$ in terms of the creation and annihilation operators b_r^*, b_r, also that $[b_r, [b_s^*, b_t]]=\delta_{rs}b_t$ and $[b_r, [b_s, b_t]]=0$. An irreducible $(k+1)$-dimensional representation of any one b_r corresponds to a state in which j similar particles may be present with $j\leqq k$. A reducible representation in 2^{pk} dimensions is given for p different b_r. In general, a new state results from the interchange of two particles and the effect of interchange is, in general, not the usual one. A parallel development is possible for integer spin fields. *C. Strachan* (Aberdeen).

Umezawa, Hiroomi, and Kawabe, Rokuo. On the problem of gauge-invariancy and divergence in the theory of elementary particles. I. Progress Theoret. Physics **5**, 769–786 (1950).

Arguments, referred to as correspondence principle ones, and various postulates about the definition of the vacuum are used to establish certain gauge-invariant results and to reduce the order of certain divergences in the theory of various elementary particles. *C. Strachan.*

Wataghin, G. On the non local interaction and on the statistical interpretation of the cut-off operators. Nuovo Cimento (9) **10**, 500–503 (1953).

It is shown that the use of invariant form factors or cut-off operators in the covariant interaction representation with a local Hamiltonian leads to an approximate description of non-local interactions. An exact treatment of the interaction representation is incompatible with the introduction of form factors, but the proposed approximation can be valid for certain classes of problems, for instance, for the non-adiabatic treatment of collision problems. A previously proposed statistical interpretation of the form factors [Nature **142**, 393–394 (1938); C. R. Acad. Sci. Paris **207**, 358–360, 421–423 (1938)] is briefly discussed. A condition which one has to impose upon the form factors in order to obtain the usual local theory in the limit of slowly varying fields is interpreted as "an equation which determines the possible masses of the elementary particles". Examples of such mass equations will be discussed in a forthcoming paper. *E. Gora* (Providence, R. I.).

Bopp, Fritz. Statistische Untersuchung des Grundprozesses der Quantentheorie der Elementarteilchen. Z. Naturforschung **8a**, 6–13 (1953).

The quantum theory of the creation or annihilation of a particle at a point and the variation in a given time interval of the state of occupation of this point are considered and conclusions are drawn about the reversibility of quantum processes and the irreversibility of ordinary statistical processes. *C. Strachan* (Aberdeen).

Williamson, E. M. Energy in the nuclear field. Nuovo Cimento (9) **10**, 113–126 (1953).

H. T. Flint [Philos. Mag. (7) **38**, 22–32 (1947); these Rev. **9**, 167] has shown for a scalar nuclear field that a five-dimensional theory may lead to a finite or vanishing value of the total energy while the expressions for the field energy and for the interaction energy may contain infinities. The purpose of the present paper is to show that the same conclusion applies to vector, pseudovector and pseudoscalar fields in interaction with nucleons. To describe these fields, tensors of suitable rank in a five-dimensional continuum are used. The expressions for the interaction energy are obtained by generalization of the corresponding four-dimensional expressions. For the two-nucleon problem, the theory leads to an interaction energy of the usual type, where, however, infinite terms are balanced by corresponding terms in the field energy. *E. Gora.*

Novobatzky, K. F. Zur Schrödinger-Gordon-Gleichung. Ann. Physik (6) **11**, 285–292 (1953).

Arguments such as those proposed by Bohm [Physical Rev. (2) **85**, 166–179, 180–193 (1952); these Rev. **13**, 709, 710] are advanced for an interpretation of the wave equation for spin zero in terms of the classical theory of an ensemble of spinless particles. *H. C. Corben.*

Costa de Beauregard, Olivier. Sur l'introduction de la théorie du photon de M. L. de Broglie dans l'électromagnétisme quantique de Schwinger. C. R. Acad. Sci. Paris **236**, 2215–2217 (1953).

Svenonius, Per, and Waller, Ivar. On the role of spin in the non-relativistic theory of the scattering of radiation. Ark. Fys. **6**, 119–121 (1953).

Pauli's non-relativistic Hamiltonian of an electron with spin [Handbuch der Physik, Bd. XXIV, T. 1, Springer,

Berlin, 1933, p. 239] is used to show that one obtains the correct expressions from the corresponding expressions in the non-relativistic theory without spin by the substitution $p \to p + i\frac{1}{2}\hbar\sigma \times k$ (p momentum operator, σ Pauli's spin matrix, k photon wave number) in every matrix element of the operator p. This result is shown to be valid for photon scattering, spontaneous emission, and many-electron problems of the type encountered in dispersion theory.

E. Gora (Providence, R. I.).

Petiau, Gérard. Sur la représentation par des fonctions sphériques des solutions des équations d'ondes des corpuscules à spin dans les potentiels constants. Cas des corpuscules de spin $\hbar/2$ et $\hbar$. C. R. Acad. Sci. Paris **236**, 1750–1753 (1953).

Šapiro, I. S. On transformation properties of wave functions of particles with spin 1/2. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **23**, 412–416 (1952). (Russian)

Under a change of sign of the time coordinate, it is possible to transform the four components of the Dirac wave function in different ways so that the sign of either the energy or the charge changes. The author discusses the meaning of such a "time reflection". The transformation under which the sign of the charge changes is non-linear. However, L. Biedenharn [Physical Rev. (2) **82**, 100 (1951); these Rev. **12**, 658] has brought about linearity in the field-free case by introducing an eight-component wave function which satisfies a generalization of the Dirac equation. According to the author, this procedure fails if a field is present. He therefore proposes a generalization involving a six-dimensional space.

N. Rosen (Haifa).

Wilker, P., et Mercier, A. Remarques sur la singularité du temps, l'utilisation d'un formalisme quantique homogène et sur la relation d'incertitude entre le temps et "l'énergie". Helvetica Phys. Acta **26**, 181–190 (1953).

Two methods are presented for writing the Schrödinger equation in a form more symmetrical between time and space coordinates. The meaning of a transformed $\psi^*\psi$ is then discussed when an arbitrary coordinate assumes the role usually played by the time.

H. C. Corben.

Husimi, Kôdi. Miscellanea in elementary quantum mechanics. I. Progress Theoret. Physics **9**, 238–244 (1953).

The density matrix of quantum statistical mechanics is applied to standard problems in non-relativistic quantum theory, a procedure that could be useful when teaching a course in quantum mechanics.

H. C. Corben.

Nakano, Huzio. An algebraic treatment of the many electron problem. Progress Theoret. Physics **9**, 33–73 (1953).

The exchange part of the interaction hamiltonian of a many-electron system is studied in the framework of second quantization, applying the representation theory of algebras to the algebra of exchange operators. The method was first proposed by H. Ostertag [Z. Physik **106**, 329–342 (1937)] and is here worked out in more complete form. The cases of 2 to 7 singly occupied electron levels are discussed in detail as an application.

L. Van Hove (Princeton, N. J.).

Petiau, Gérard. Sur le calcul de la diffusion des corpuscules de spin $\hbar/2$ par un potentiel pseudoscalaire coulombien. C. R. Acad. Sci. Paris **236**, 2303–2305 (1953).

Flügge, S., und Woeste, K. Der Atomkern als kompressibler Tropfen. I. Der Atomkern im Grundzustand. Z. Physik **132**, 384–398 (1952).

There is evidence against the conception of incompressibility in the liquid drop model of the atomic nucleus. In the energy of the ground state is included, in addition to the usual terms, a term depending on compressibility. A variation principle for the energy yields equations with an artificial singularity at the boundary of the nucleus. These are integrated, for the interior of the nucleus, ignoring the boundary singularity and give neutron and proton densities increasing towards the boundary. For $Z>36$, approximately, the pressure is negative at the centre of the nucleus. The average pressure vanishes at about $Z=91$. The radius of the nucleus increases more rapidly than $A^{\frac{1}{3}}$. Binding energies are calculated.

C. Strachan (Aberdeen).

Woeste, K. Der Atomkern als kompressibler Tropfen. II. Der schwingende Kerntropfen. Z. Physik **133**, 370–393 (1952).

The proper vibrations of a compressible liquid-drop nucleus are investigated. Viscosity, transverse waves, and nuclear rotation are excluded from consideration. First the ratio of proton density to neutron density is taken as constant. Deformations are assumed to have axial symmetry and the motion to be irrotational. Second a two component (proton and neutron) mixed fluid is discussed. Diagrams and numerical results are given. The eigenfrequencies are lower than those obtained for an incompressible one-component fluid drop, and the limit in Z for spontaneous fission is lower. New energy terms occur which are not present for the incompressible drop.

C. Strachan.

Olsen, H., Werenskiold, P., and Wergeland, H. Retardation of meson fields. Norske Vid. Selsk. Forh., Trondheim **25** (1952), 54–59 (1953).

The classical equation of the scalar meson field generated by a point source of given trajectory is solved and the solution used for a brief discussion of retardation effects.

L. Van Hove (Princeton, N. J.).

Thermodynamics, Statistical Mechanics

Fényes, Imre. Ergänzungen zur axiomatischen Begründung der Thermodynamik. I. Eine axiomatische Deutung des Begriffes "Intensitätsparameter". Z. Physik **134**, 95–100 (1952).

The author re-interprets, in a very general manner, Carathéodory's axioms for temperature.

C. C. Torrance.

Popoff, Kyrille. Sur la thermodynamique des processus irréversibles. Z. Angew. Math. Physik **3**, 440–448 (1952).

It is shown that L_{α} is symmetric to the first approximation when ΔS is arbitrary.

C. C. Torrance.

San Juan, Ricardo. An application of vector spaces to the phase rule of thermodynamics. Univ. Lisboa. Revista Fac. Ci. B. (2) **1**, 107–112 (1952). (Spanish)

By using coordinates proportional to masses (rather than to numbers of atoms), it is possible to avoid an incompatibility between Volterra's rule and the phase rule.

C. C. Torrance (Monterey, Calif.).

Green, Melville S. Markoff random processes and the statistical mechanics of time-dependent phenomena. J. Chem. Phys. **20**, 1281–1295 (1952).

This paper is an attempt to develop some general principles for solving non-equilibrium problems which involve systems of large numbers of molecules. The basic postulate is that one can find a number of gross variables which are macroscopic and whose probability distribution is generated as a stationary Markoff process. An example of such gross variables in a fluid might be the momentum, energy, and number of particles in volume elements of the order of one cubic millimeter into which the fluid might be decomposed. A Fokker-Planck type of equation is derived whose solution gives the variation of the distribution function of the gross variables with time. The Onsager reciprocal relations [Physical Rev. (2) **37**, 405–426 (1931)] between phenomenological constants of coupled irreversible processes are a direct consequence of the theory developed in the paper.

E. W. Montroll (College Park, Md.).

Hashitsume, Natsuki. A statistical theory of linear dissipative systems. Progress Theoret. Physics **8**, 461–478 (1952).

This paper is concerned with the development of statistical mechanics of systems slightly removed from equilibrium. A set of gross macroscopic variables is introduced and it is postulated that the time rate of change of the entropy of a system is a linear combination of the rates at which the gross variables out of equilibrium are changing. By combining the above postulate with the Onsager relations a set of equations of the Fokker-Planck type are derived for the variation of the distribution functions of the gross variables with time. These equations are essentially the same as those derived by M. S. Green [see the preceding review]. The two special problems investigated by the new formalism are the Brownian motion of a colloidal particle and the fluctuations in heat flow in an isolated rod.

E. W. Montroll (College Park, Md.).

Bazarov, I. P. The dynamical equation of Gibbs, the kinetic equation of Boltzmann and irreversibility. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1952**, no. 2, 75–78 (1952). (Russian)

The author indicates how in deriving Boltzmann-like equations in statistical mechanics the use of certain approximations can lead to irreversible processes, even though the original dynamical equations are reversible, and can even lead to equations which imply decrease of entropy.

W. Kaplan (Zurich).

Nishiyama, Toshiyuki. A quantum theory of boson assemblies. II. A classical approach to many-boson problems. Progress Theoret. Physics **9**, 245–267 (1953).

The medium-like behavior of a system of bosons is investigated in the limit of high densities. The classical density and velocity functions are defined as the expectation values of the corresponding quantum operators for Gaussian wave packets. The conservation equations for classical density, momentum and energy are obtained, neglecting fluctuations, from the equations for the corresponding operators, derived by the author in part I [same journal **8**, 655–668 (1952); these Rev. **14**, 710]. The collective excitations of the system are analysed in terms of sound waves, and the interaction between individual particles and the sound wave field is discussed.

L. Van Hove (Princeton, N. J.).

Green, H. S. Boltzmann's equation in quantum mechanics. Proc. Phys. Soc. Sect. A. **66**, 325–332 (1953).

The standard procedure to correct for quantum effects in the Boltzmann equation of gas theory consists in introducing the quantum-mechanical cross-section in the collision integral. The present paper gives a purely quantum-mechanical derivation of Boltzmann's equation, based on the assumptions that all but binary encounters can be neglected and that before a two-particle collision the two particles are statistically independent. The last assumption introduces irreversibility. The transport equation obtained differs from the conventional quantum form of the Boltzmann equation [Chapman and Cowling, The mathematical theory of non-uniform gases, Cambridge, 1939; these Rev. **1**, 187] by two correction terms proportional to the square of the number density, one classical and one quantum-mechanical.

L. Van Hove (Princeton, N. J.).

Bernard, J.-J., et Siestrunck, R. Aspect cinétique des mouvements fluides unidimensionnels. Recherche Aéronautique no. **31**, 45–48 (1953).

After discussing some past work and alternative procedures in the so far unsuccessful attempts to construct a satisfactory kinetic theory of plane shock-waves, the authors propose to consider the phenomenon as a diffusion process. The components are taken as composed of molecules whose distribution yields the two end-states and the various intermediates. The authors first set up some simple heuristic formulae indicating that the effect of such a diffusion process is formally the same as that of viscosity and thermal conduction. These formulae employ a spectral resolution of the distribution function. Then they attempt to incorporate this idea in a solution of the Boltzmann equation. They adopt a scheme analogous to Enskog's, where certain nonlinear terms are neglected. They consider only a "second approximation" near equilibrium. For purposes of calculation they replace the continuous spectrum by a discrete one composed of n parts. When $n=2$ and the components correspond to the two end-states, their results reduce to these of Mott-Smith [Physical Rev. (2) **82**, 885–892 (1951); these Rev. **12**, 891]. Considering this to be insufficient, they take $n=3$, letting the third component be the equilibrium state corresponding to the arithmetic mean of the end velocities. Their results are expressions for heat flux and pressure and an indeterminate temperature-velocity equation. Their conclusion is: "La théorie cinétique des gaz apporte ainsi à l'aérodynamique de l'onde de choc des généralisations permettant d'améliorer la connaissance de sa structure sans entraîner les développements formellement compliqués que comporte généralement la fonction de distribution; l'application de cette théorie est, en tout rigueur, limitée aux écoulements transsoniques, mais l'extrapolation à de plus grandes vitesses n'entraîne aucune divergence et permet un premier pas vers la théorie moléculaire des écoulements supersoniques." In view of the great difference between their results when $n=2$ and $n=3$, and the indefinite number of arbitrary quantites which the extension of their procedure would introduce, not to mention its diversity from other methods now being proposed, some readers may question whether the kinetic theory in its present state can yield any definite and reliable information regarding the structure of shocks.

C. Truesdell (Bloomington, Ind.).

anics.

ffects
ntro-
lision
tum-
ed on
n be
two
ption
ained
Boltz-
atical
Rev.
quare
tum-

J.).

des
Aéro-

oced-
uct a
thors
ocess.
whose
inter-
ristic
ocess
ermal
on of
orate
They
non-
econd
cula-
e one
corre-
ese of
these
take
state
cities.
e and
con-
nsi à
per-
sans
s que
appli-
coule-
andes
emier
uper-
their
ber of
edure
other
estion
d any
ure of

d.).